Dynamics of star clusters containing stellar mass black holes: 1. Introduction to Gravitational Waves

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Bonn

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Outline

- What are the gravitational waves?
- Generation of gravitational waves
- Detection methods
- Astrophysical Sources
- Detected GW Sources

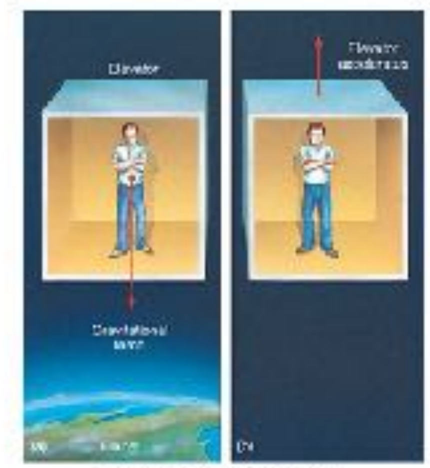
Principle of Equivalence

- Principle of Equivalence: inertial mass = gravitational mass

$$\mathbf{F} = m_i \mathbf{a} = m_g \mathbf{g}$$

$$\mathbf{g} = -\frac{GM}{r^3}\mathbf{r}$$

Figure 13.15
Einstein's Elevator



- The trajectory of a particle in gravity does not depend on the mass: geometrical nature
- In a freely falling frame, one cannot feel the gravity.
- However, the presence of the gravity will cause tidal force. Again, this is similar to the difference between flat and curved space.

Curved Spacetime

- General geometry of space-time can be characterized by metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- In the absence of gravity, flat spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$g_{\mu\nu} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The presence of gravity causes deviation from flat spacetime: curved spacetime

Tidal gravitational forces

- Let the positions of the two nearby particles be ***x*** and ***x***+***χ***, then the equations of motions are

$$\frac{d^2 x^i}{dt^2} = -\delta^{ij}\frac{\partial \Phi(\mathbf{x})}{\partial x^j}\,, \qquad \frac{d^2 (x^i + \chi^i)}{dt^2} = -\delta^{ij}\frac{\partial \Phi(\mathbf{x}+\chi)}{\partial x^j}$$

- If ***χ*** is small, one can expand **Φ**

$$\frac{\partial \Phi(\mathbf{x}+\chi)}{\partial x^j} \approx \frac{\partial \Phi(\mathbf{x})}{\partial x^j} + \frac{\partial}{\partial x^k}\left(\frac{\partial \Phi(\mathbf{x})}{\partial x^j}\right)\chi^k + \ldots$$

- Then the relative acceletation between two particles becomes

$$\frac{d^2 \chi^i}{dt^2} = -\delta^{ij}\left(\frac{\partial^2 \Phi}{\partial x^j \partial x^k}\right)_{\mathbf{x}}\chi^k$$

Tidal acceleration tensor

In GR, similar expression can be derived

- "Geodesic deviation" equation

$$\frac{D^2\chi^\lambda}{D\tau^2} = R^\lambda_{\nu\mu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\chi^\mu$$

Riemann curvature tensor

- In the weak field limit, $v \ll c$, $|\boldsymbol{\Phi}| \ll c^2$. The only remaining components are $\mu=\nu=0$. Therefore,

$$\frac{d^2\chi^i}{dt^2} = R^i_{0j0}\chi^j$$

i.e., $$R^i_{0j0} = -\frac{\partial^2\Phi}{\partial x^i \partial x^j}$$

- Ricci tensor can be defined by contraction of Riemann tensor:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$$

Einstein's Field Equation

- In Newtonian dynamics, the motion of a particle is governed by the gravitational potential which can be computed by the Poisson's equation:

$$\nabla^2\Phi = \delta^{ij}\left(\frac{\partial^2\Phi}{\partial x^i\partial x^j}\right) = 4\pi G\rho$$

i.e., summation over the quantities that describe the geodesic deviation (=tidal acceleration tensor).

- The Ricci tensor is similarly defined with Φ, i.e., summation of all tidal components.
- In vacuum, $R_{\mu\nu} = 0, \text{ similar to } \nabla^2\Phi = 0$
- In the presence of energy (mass, etc.), Einstein's field equation becomes

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}$$

Energy-momentum tensor

Curvature scalar $R = R^\mu_\mu$

Field equation in weak field limit

flat part — small perturbation

- In the weak field limit, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$
- Einstein's field equation becomes

$$\Box h_{\mu\nu} = \frac{\partial^2}{\partial x^\lambda \partial x^\mu} h^\lambda_\nu - \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^\lambda_\mu + \frac{\partial^2}{\partial x^\mu \partial x^\nu} h^\lambda_\lambda = -16\pi G S_{\mu\nu}$$

where $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda_\lambda$ and $\Box = -\frac{\partial^2}{\partial t^2} + \nabla^2$

- Perform coordinate transformation,

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu$$

- Then, $h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \xi_\mu}{\partial x^\nu} - \frac{\partial \xi_\nu}{\partial x^\mu}$

Choice of guage (i.e., coordinates)

- Define a trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$$

and impose the Lorentz guage condition $\frac{\partial \bar{h}_{\mu\nu}}{\partial x_\nu} = 0$

then equation for $\bar{h}_{\mu\nu}$ becomes

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Further Properties

- In vacuum (i.e., $T_{\mu\nu}$=0),

$$\Box \bar{h}_{\mu\nu} = 0 \quad \text{i.e, equation for plane waves}$$

- The solution can be written in the form

$$\bar{h}_{\mu\nu} = Re\{A_{\mu\nu} \exp(ik_\lambda x^\lambda)\}$$

- One can choose a coordinate system by rotating so that

$$A^\mu_\mu = 0 \text{ (traceless)}, \quad A_{0_i} = 0 \text{ purely spatial}$$

- The wave is transverse in Lorentz gauge. Therefore the metric perturbation becomes very simple in transverse-traceless (TT) gauge. Also $\bar{h}_{\mu\nu} = h_{\mu\nu}$ in TT coordinates.

Effects of GWs in TT gauge

- In TT guage, GWs traveling along z-direction can be written with only two components, h_+ and h_x: they are called 'plus' and 'cross' polarizations

$$h_{\mu\nu}^{TT} = h_+(t-z)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + h_\times(t-z)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

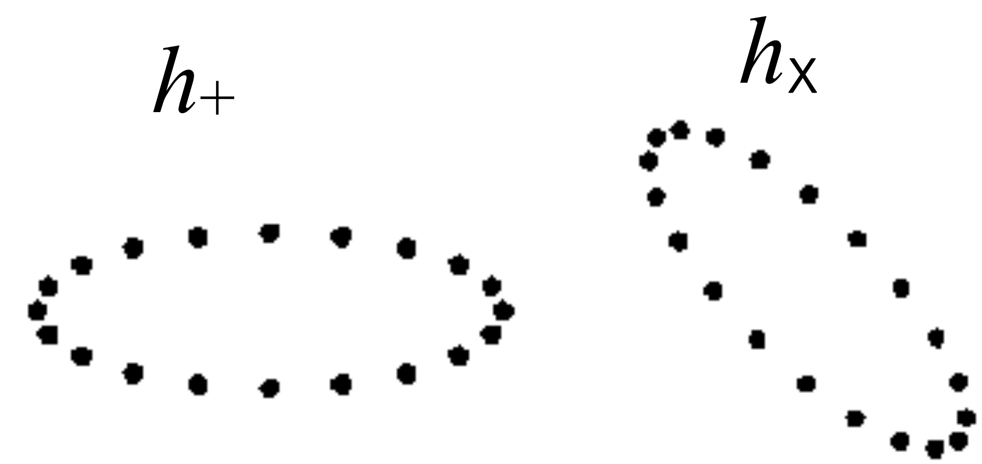
- In these coordinates, the line element becomes

$$ds^2 = -dt^2 + (1+h_+)dx^2 + (1-h_+)dy^2 + dz^2 + 2h_\times dxdy$$

- The lengths in x- and y- directions for h_+ then oscillate in the following manner

$$L_x = \int_{x_1}^{x_2} \sqrt{1+h_+}dx \approx (1+\frac{1}{2}h_+)L_{x0};$$

$$L_y = \int_{y_1}^{y_2} \sqrt{1-h_+}dy \approx (1-\frac{1}{2}h_+)L_{y0}$$



Generation of gravitational waves

- The wave equation $\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$

  gives formal solution of

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4 \int \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x \approx \frac{4}{r} \int T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}') d^3x$$

- One can show that

$$\bar{h}_{jk}(t, \mathbf{x}) = \frac{2}{r} \frac{d^2 I_{jk}(t-r)}{dt^2}$$

where $I_{jk} = \int \rho x^j x^k d^3x$

- Using the identities

$$\frac{\partial T^{tt}}{\partial t} + \frac{\partial T^{kt}}{\partial x^k} = 0, \text{ and } \frac{\partial^2 T^{tt}}{\partial t^2} = \frac{\partial^2 T^{kl}}{\partial x^k \partial x^l}$$

Order of magnitude estimates of GW amplitude

- Note: the moment of inertia tensor is not exactly the same as the quadrupole moment tensor, but in TT guage, it does not matter.

$$I_{jk} = \int \rho x^j x^k d^3x \qquad Q_{jk} = \int \rho \left(x^j x^k - \frac{1}{3} r^2 \delta_{jk} \right) d^3x$$

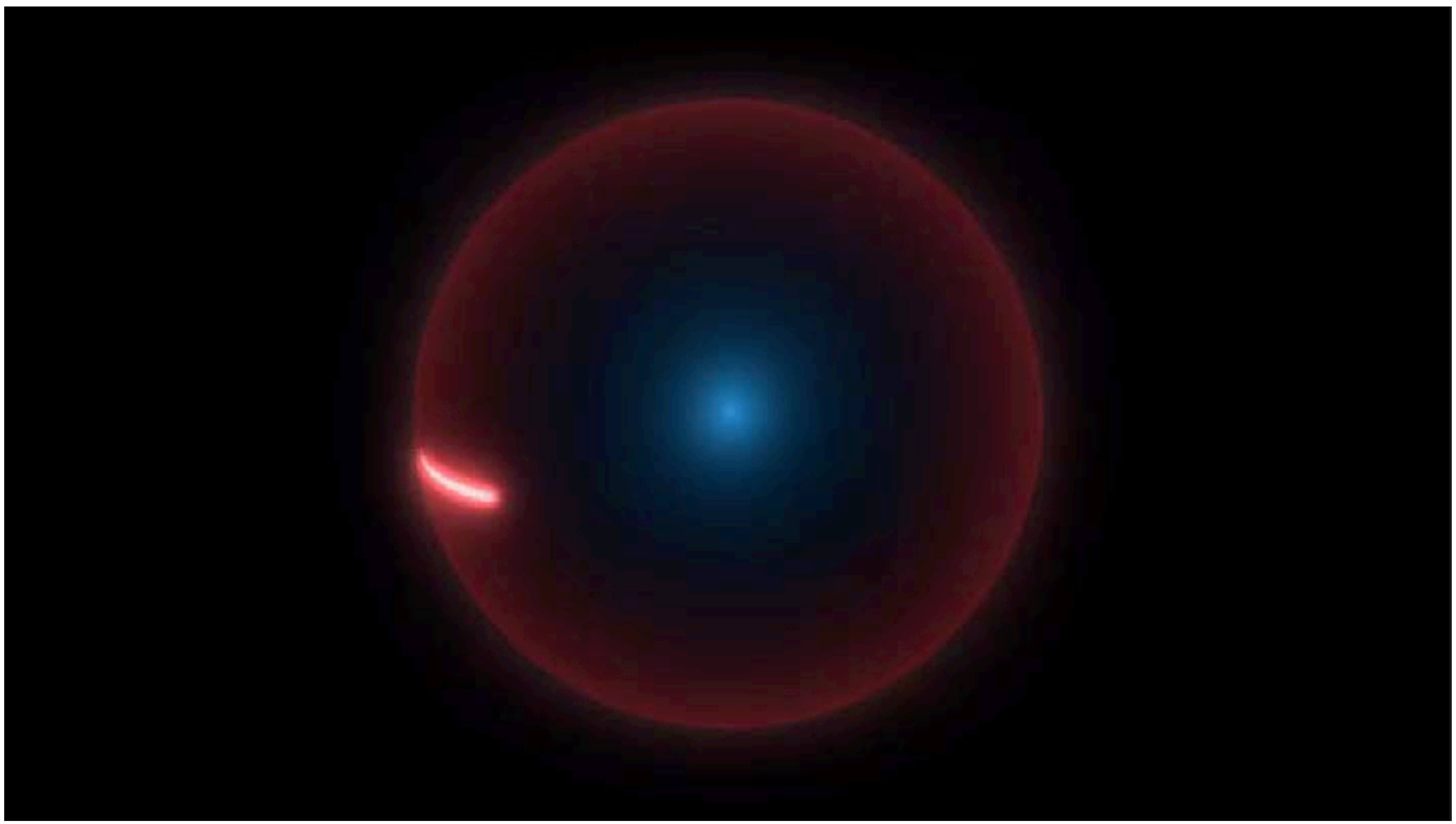
- Quadrupole kinetic energy

$$\ddot{I}_{jk} \sim \frac{(\text{mass})\times(\text{size})^2}{(\text{transit time})^2} \sim \text{quadrupole Kinetic E.} = \epsilon M c^2$$

- In most cases, $\boldsymbol{\epsilon}$ is small, but it could become ~0.1

$$h_{jk} \sim 10^{-22} \left(\frac{\epsilon}{0.1}\right) \left(\frac{M}{M_\odot}\right) \left(\frac{100\text{Mpc}}{r}\right)$$

How small is $h \sim 10^{-21}$ which was the detected amplitude of the GW150914?



Simulation created by T. Pyle, Caltech/MIT/LIGO Lab

Measurement of GWs: 1. Tidal forces

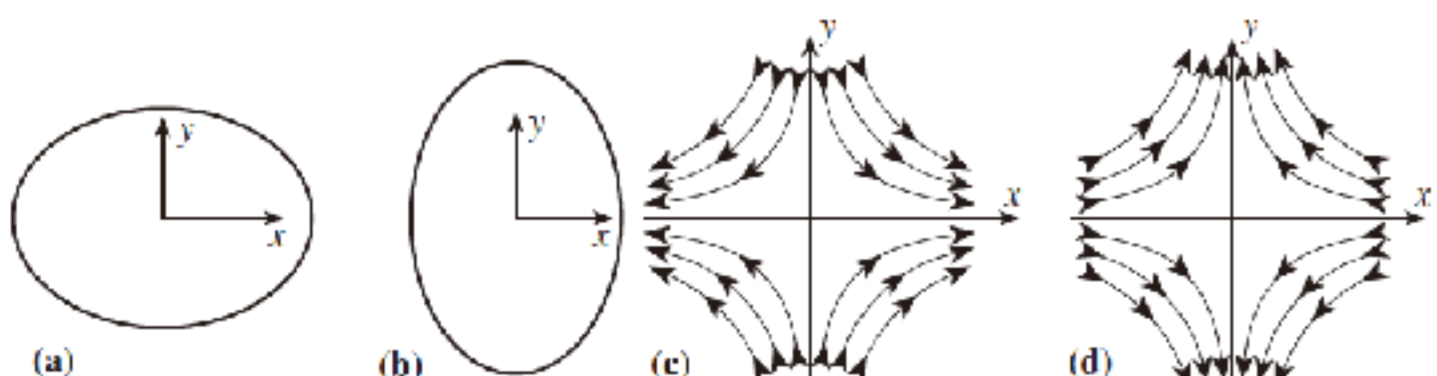
- Equation of geodesic deviation becomes GW tidal acceleration:

$$\frac{d^2\delta x^j}{dt^2} = -R_{j0k0}x^k = \frac{1}{2}\ddot{h}_{jk}x^k$$

- Riemann tensor R_{j0k0} is a gravity gradient tensor in Newtonian limit

$$R_{j0k0} = -\frac{\partial^2\Phi}{\partial x^j \partial x^k} = -\omega_{GW}^2 h_{jk}$$

- Tidal force can cause resonant motion of a metallic bar = bar detector

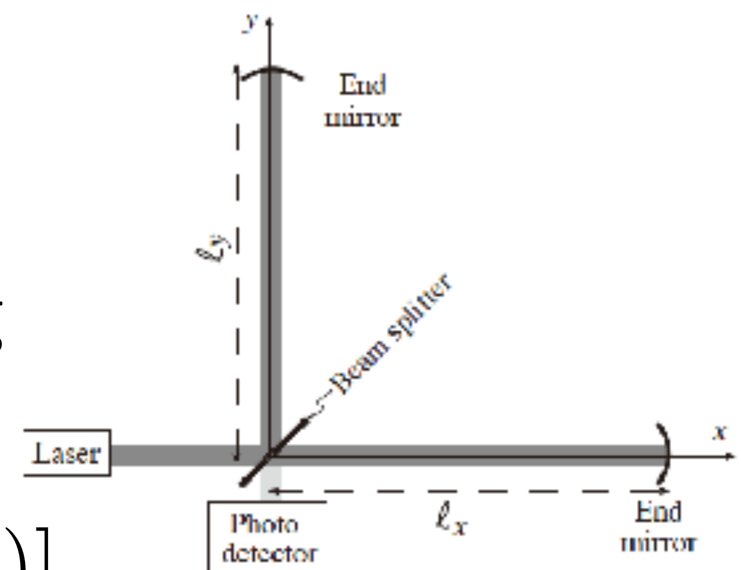


Force field

- Gravity gradiometer can be used as a gravitational wave detector

Measurement of GWs: 2. Laser Interferometer

- Consider a simple Michelson interferometer with $l_x \approx l_y \approx l$.



- The phase difference of returning lights reflected by x and y ends

$$\Delta\phi = \phi_x - \phi_y \approx 2\omega_0 \left[l_x - l_y + l h(t) \right]$$

- Toward the laser: $E_{laser} \propto e^{i\phi_x} + e^{i\phi_y}$

- Toward the photo-detector:

$$E_{PD} \propto e^{i\phi_x} - e^{i\phi_y} = e^{i\phi_y}(e^{i\Delta\phi} - 1)$$

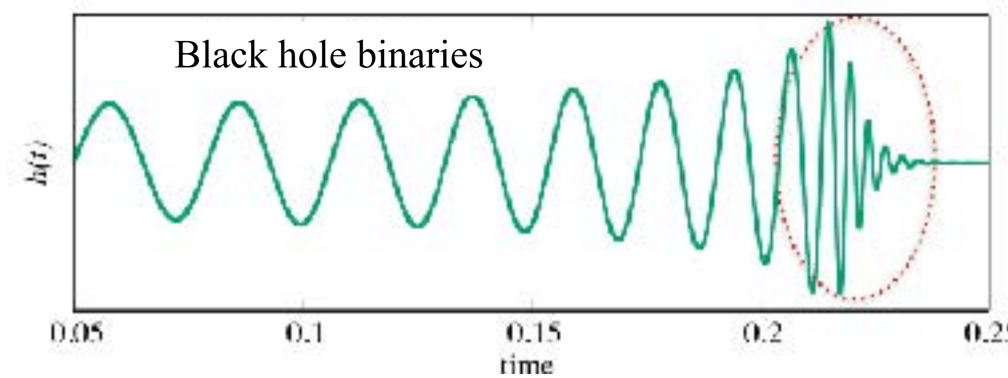
- For small $\Delta\phi$,

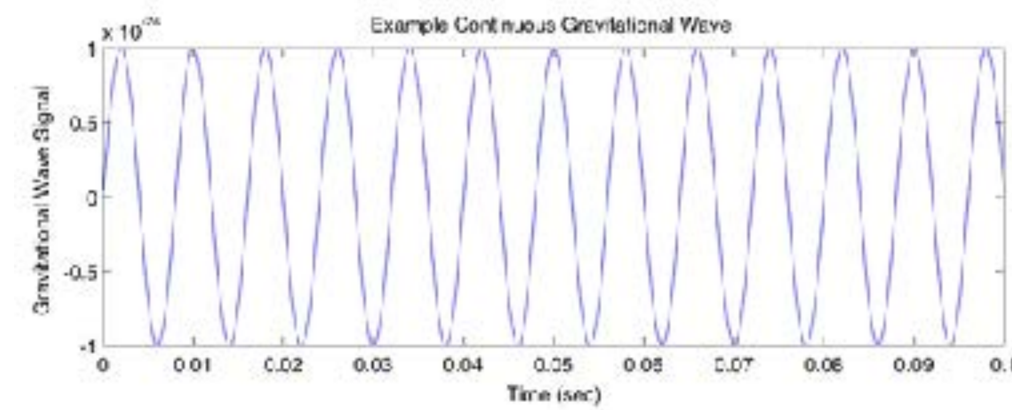
Time varying part

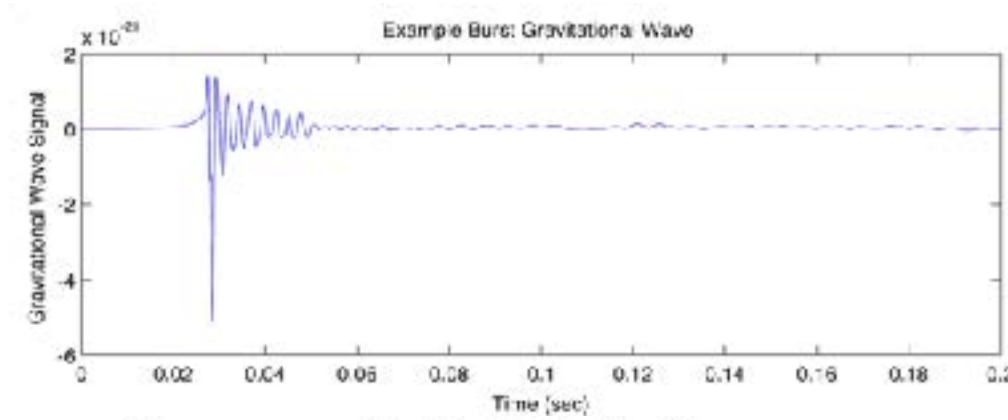
$$I_{PD} \propto |e^{i\Delta\phi} - 1|^2 \approx |\Delta\phi|^2 \approx 4\omega_0^2 (l_x - l_y)^2 + 8\omega_0^2 (l_x - l_y) l h(t)$$

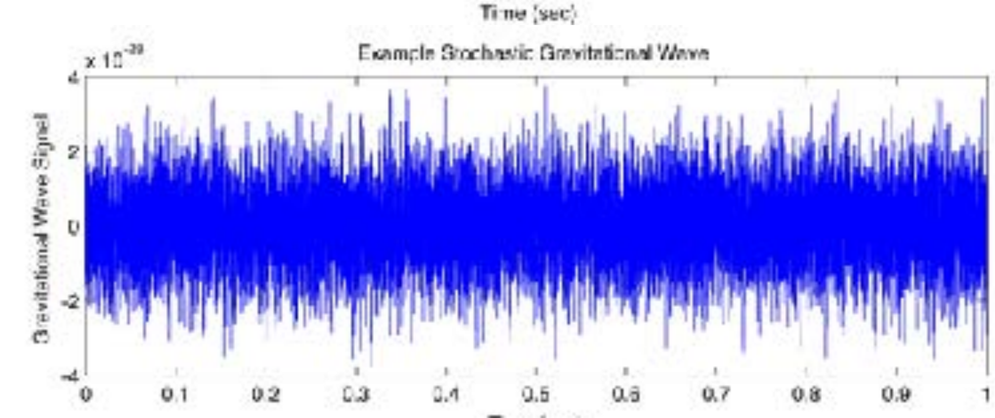
Astrophysical sources

- **Compact Binary Coalescence (neutron stars and black holes)**
  - Strong signal, but rare
  - Computable waveforms
- **Continuous (~ single neutron stars)**
  - Weak, but could be abundant
  - Deviation from axis-symmetry is not known
- **Burst (supernovae or gamma-ray bursts)**
  - GW amplitudes are now well known, not frequent $I_{jk} = \int \rho x^j x^k d^3x$
- **Stochastic**
  - Superposition of many random sources
  - Could be useful to understand distant populations of GW sources or early universe





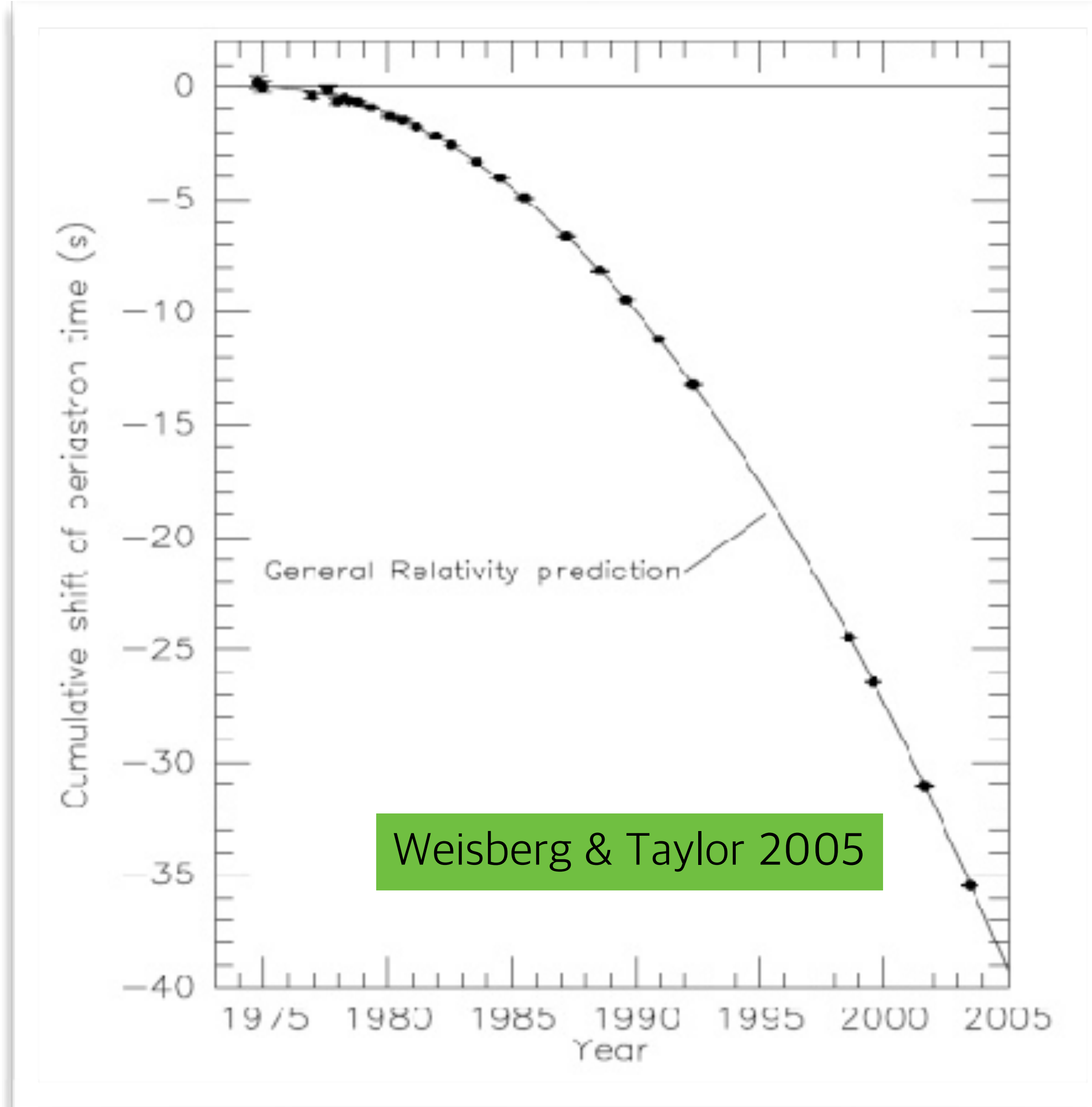


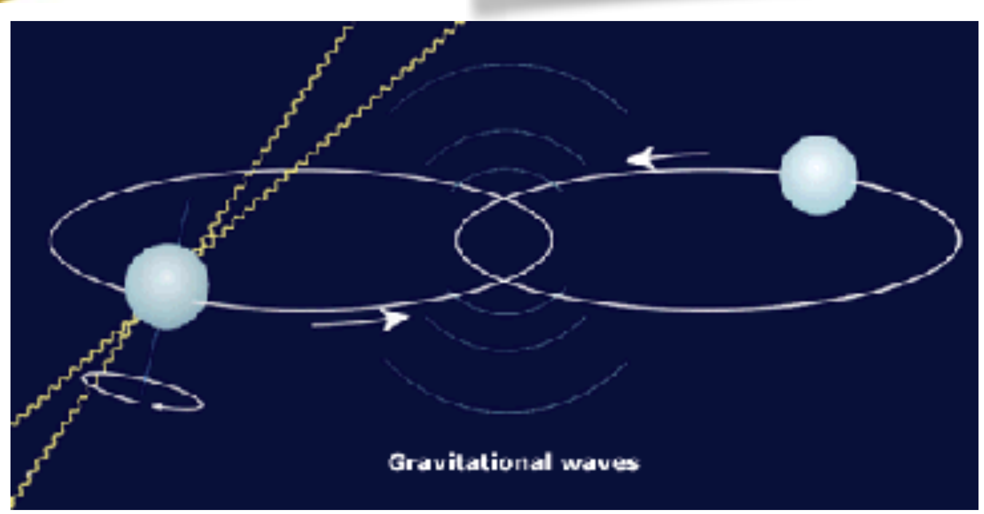


Confirmation of GW with Binary Pulsar

- Orbit of binary neutron stars shrinks slowly
- Hulse & Taylor received Nobel Prize in 1993 for the discovery of a binary pulsar

PSR 1913+16





P=7.75 hrs
a=1.6 x 10^6 km
M_1=1.4 M_{sun}
M_2=1.35 M_{sun}
Time to merge=3x10^8 yrs

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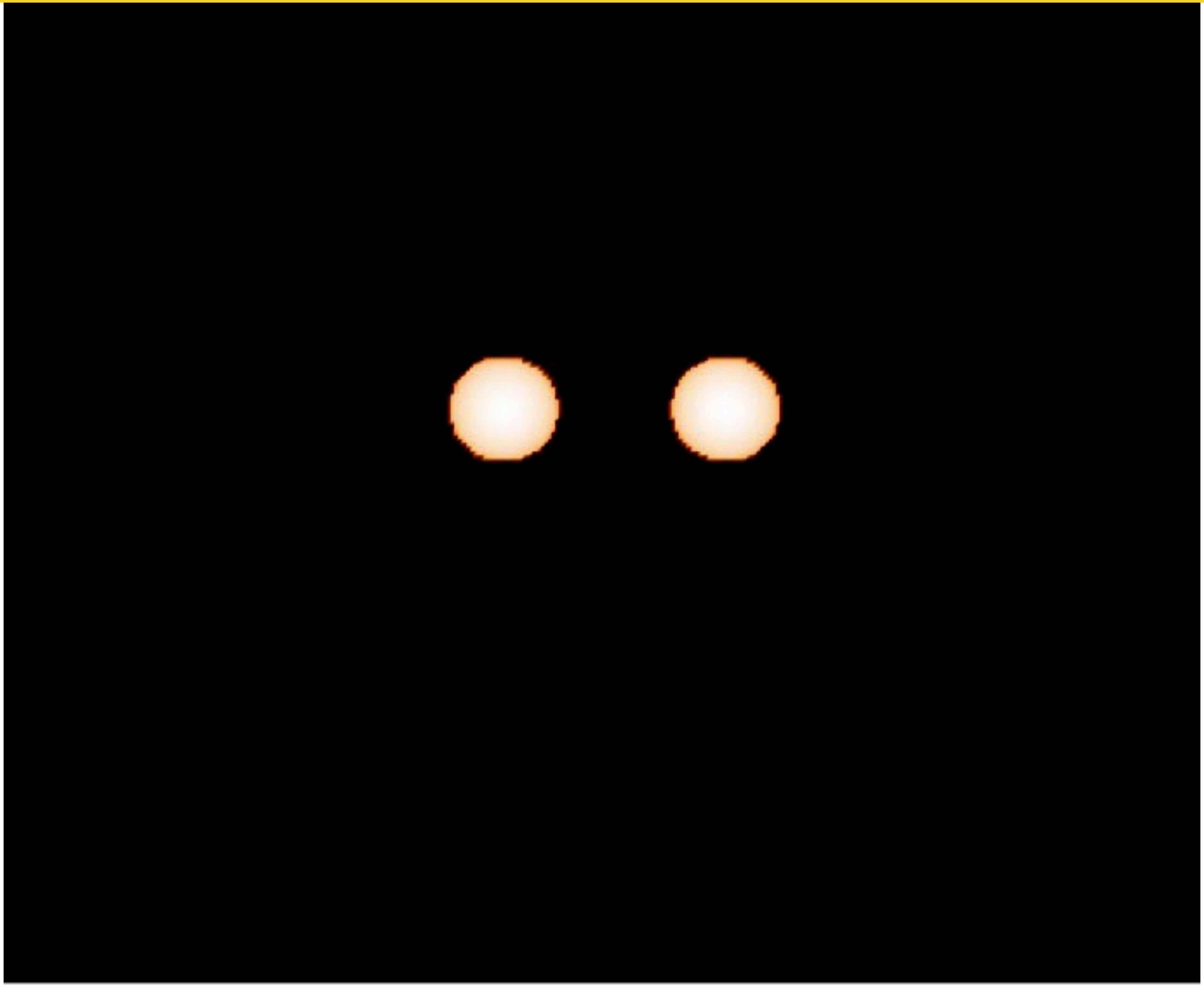
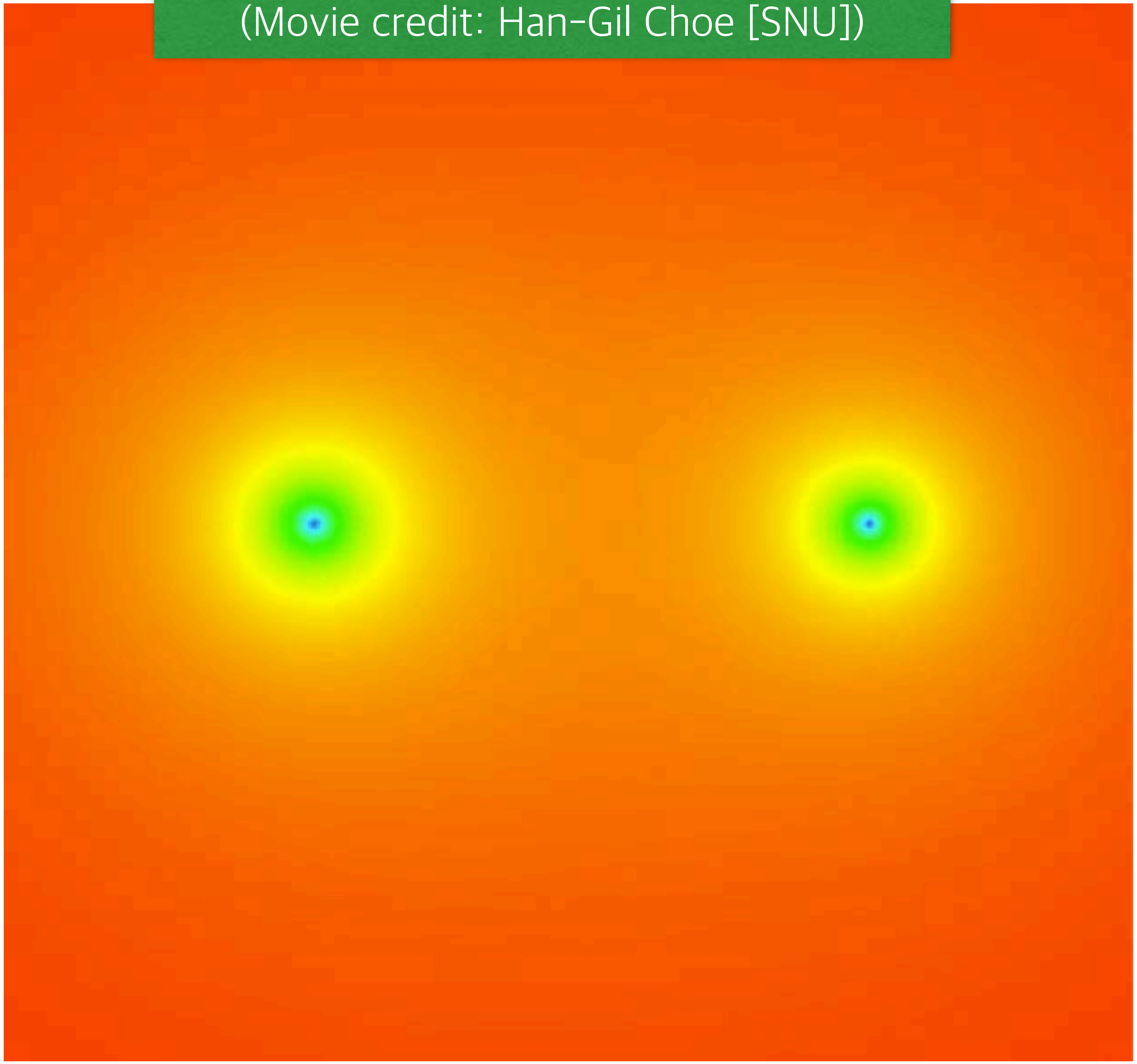
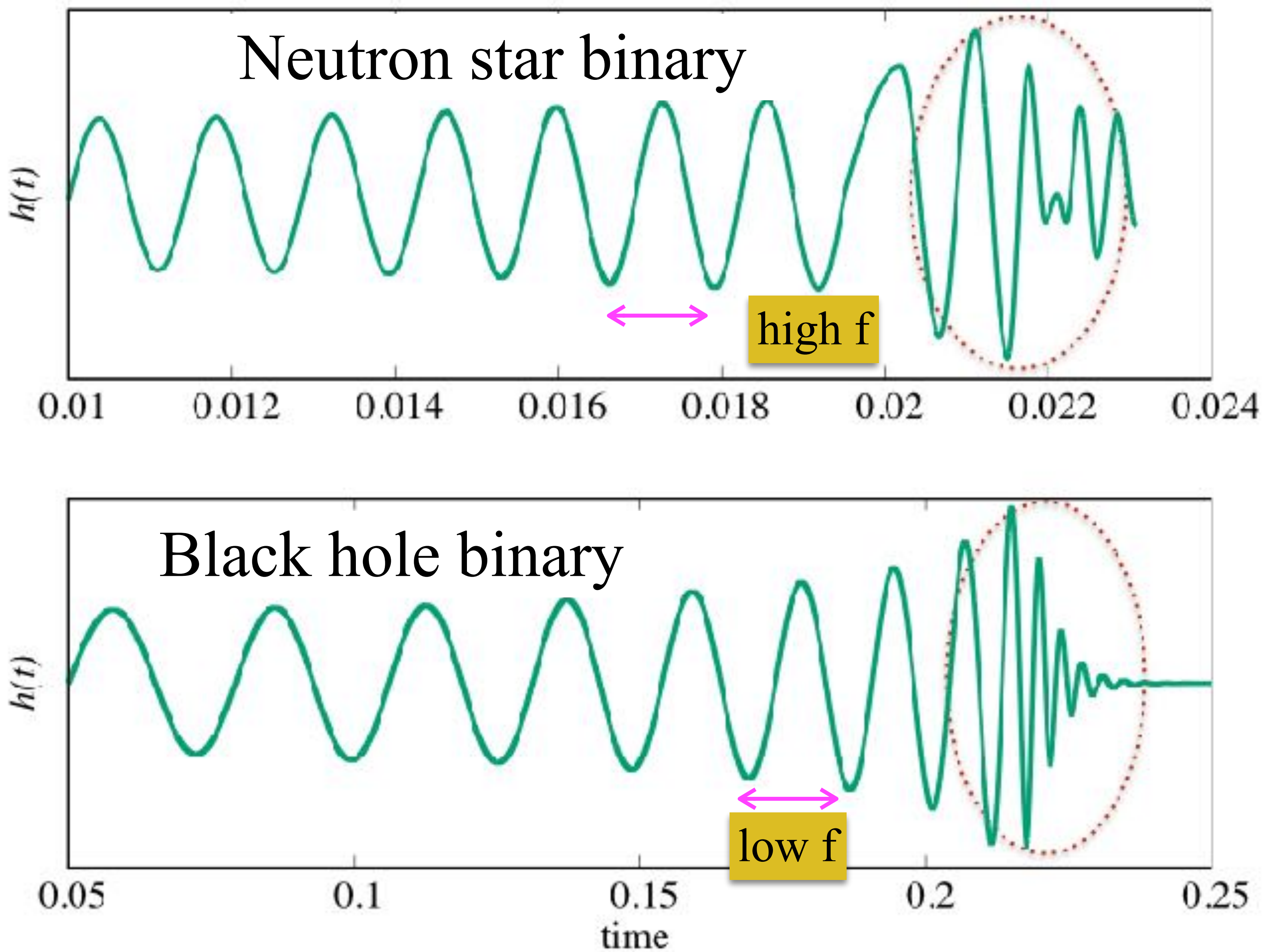
About 300 million years after, the following event is expected
(Movie credit: Gwanho Park [SNU])



The final moment of merger of black holes
(Movie credit: Han-Gil Choe [SNU])

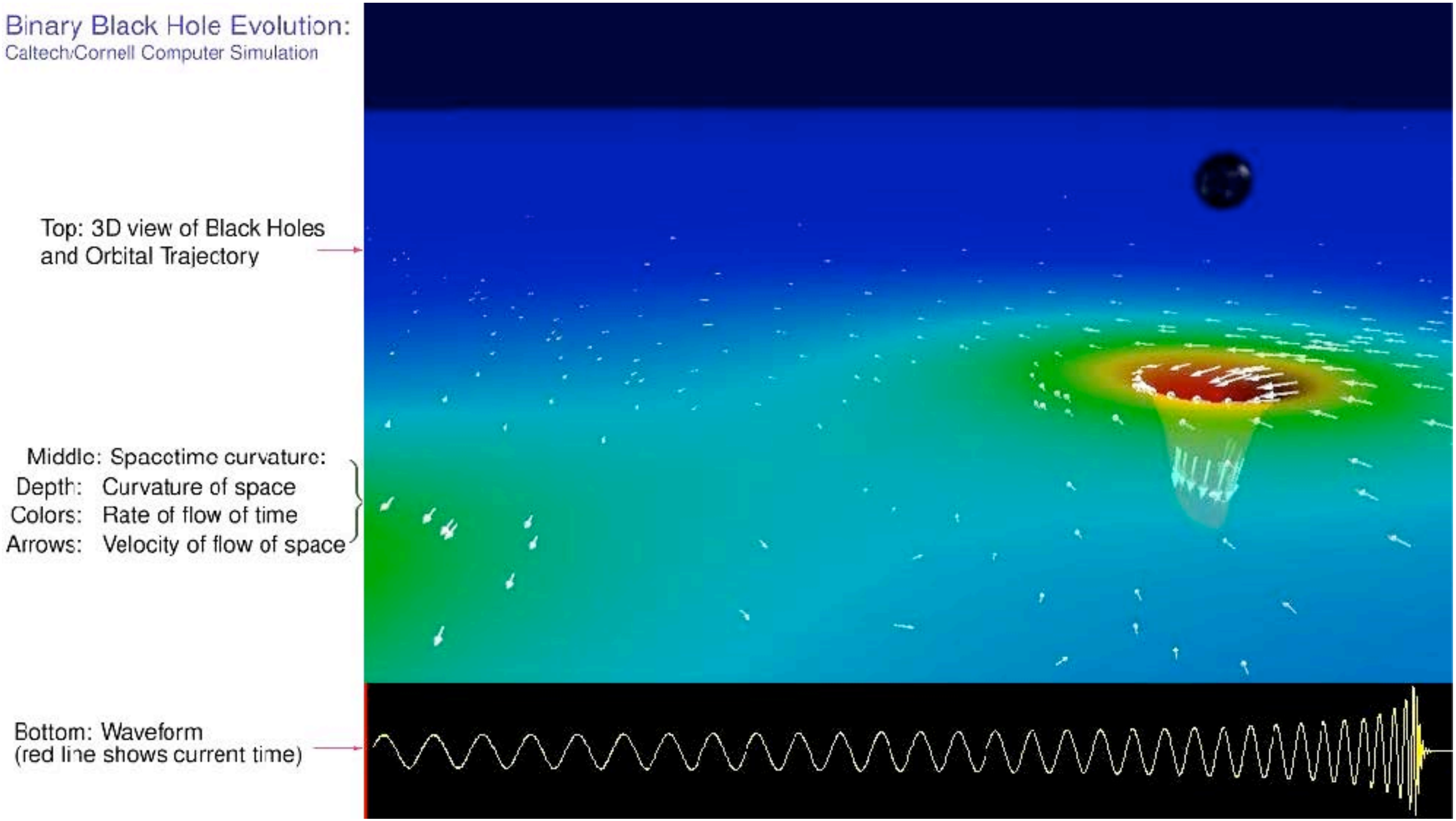


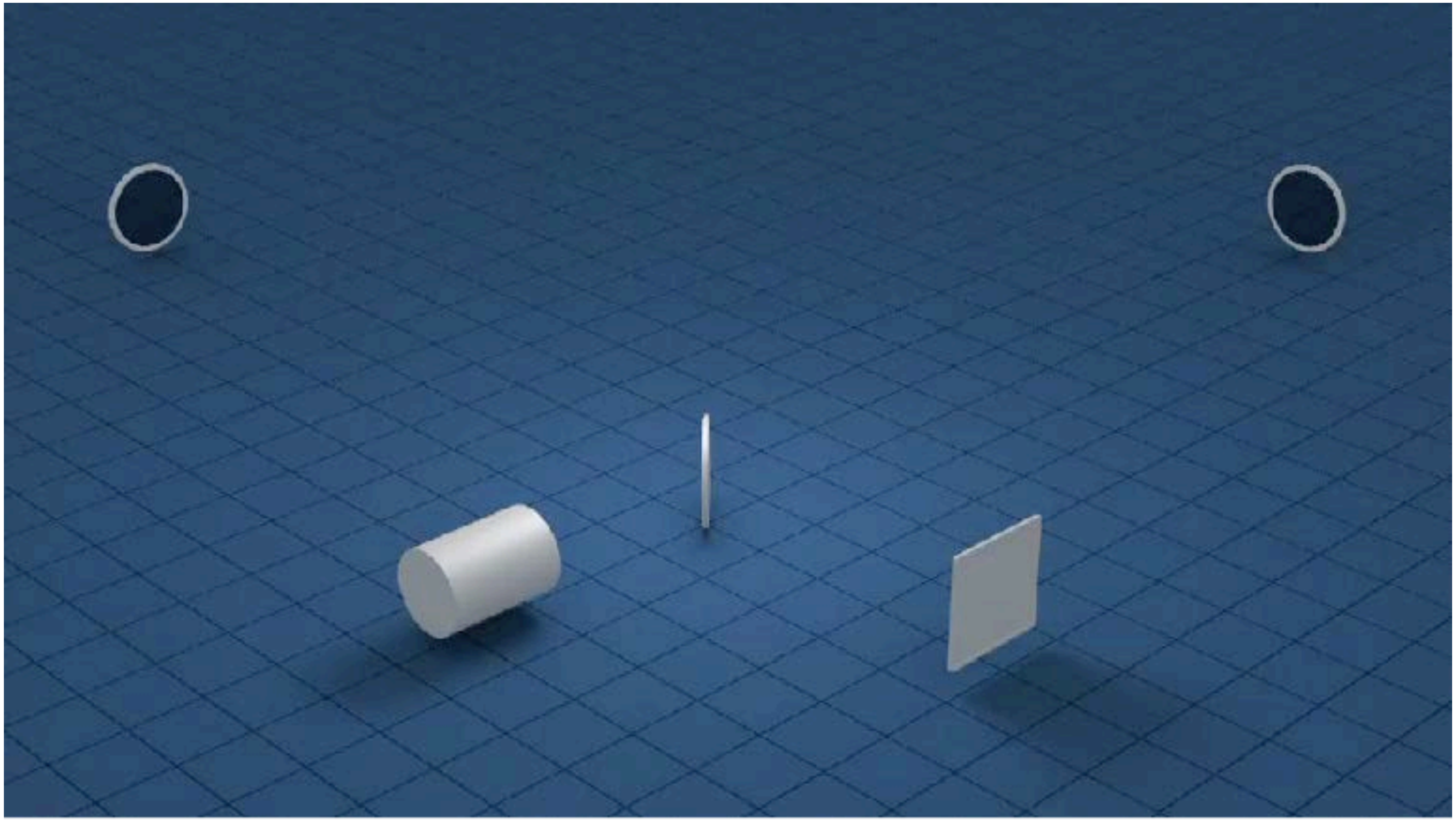
Waveform tells many things



Comparison of waveforms between NS and BH mergers: Note differences in frequencies and shape .

GW from a merging black hole

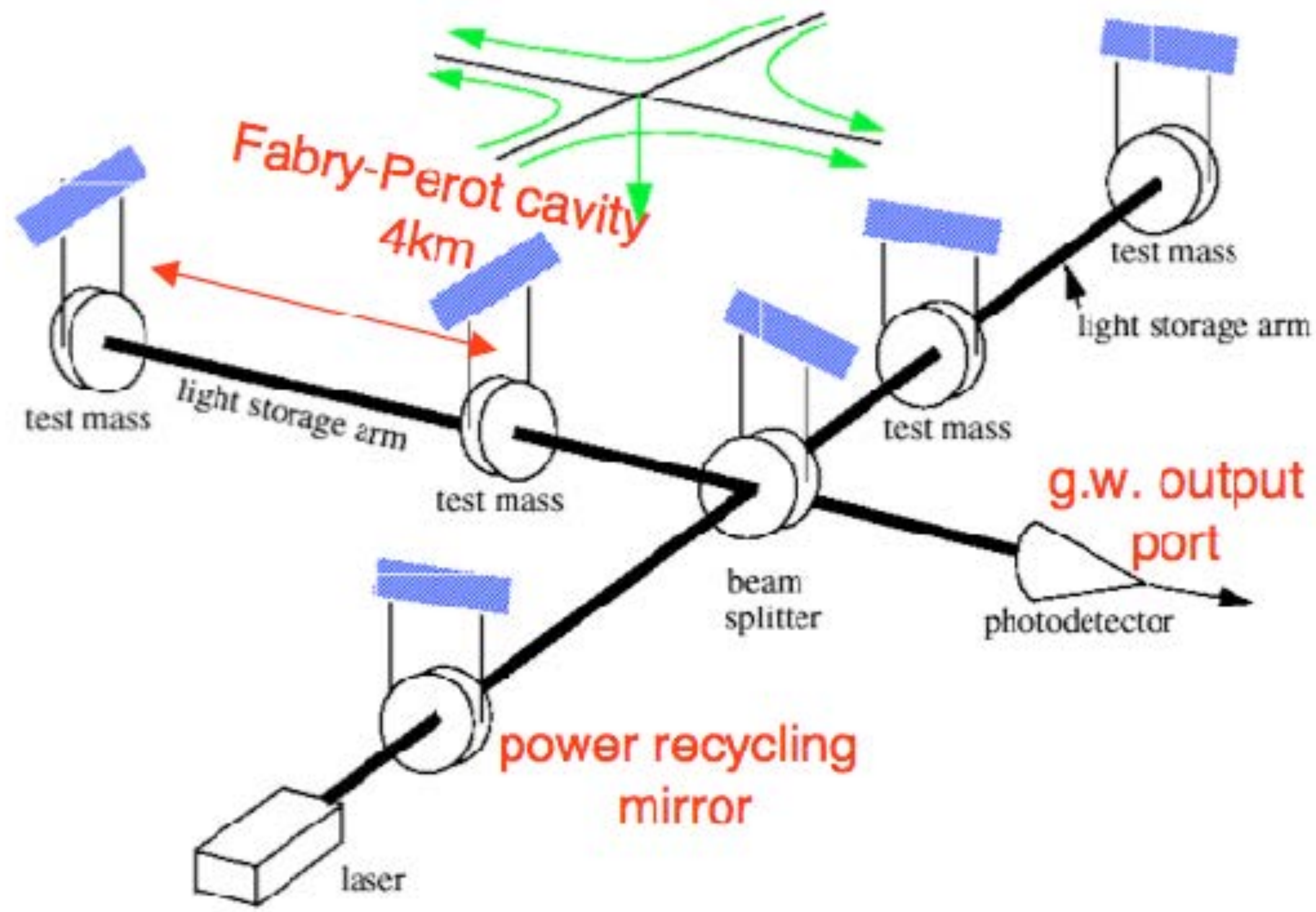




Principles of GW Detector

Image credit: LIGO/T. Pyle

Gravitational Wave Detector 1990



K. Thorne, R. Weiss, R. Drever

- 2002-2010 Initial LIGO
- 2015.9 ~ Advanced LIGO (10times better sensitivity)

Simple Estimates of Sensitivity of Interferometers

- If the length resolution is λ_{laser}, detectable strain is

$$h \equiv \frac{\Delta l}{l} = \frac{\lambda_{laser}}{l} = \frac{10^{-6}\mathrm{m}}{10^{3}\mathrm{m}} = 10^{-9}$$

- Optical path length can be significantly increased by adopting optical cavity, but should be smaller than GW wavelength (~1000 km for 300 Hz)

$$h \sim \frac{\Delta l}{l_{eff}} \sim \frac{\lambda_{laser}}{\lambda_{GW}} \sim \frac{10^{-6}\mathrm{m}}{10^{6}\mathrm{m}} = 10^{-12}$$

- However, due to quantum nature of the photons, the length resolution coud be as small as $N^{-1/2}_{photons}\lambda_{laser}$. Thus sentivity could reach

$$h \sim N_{photons}^{-1/2} \frac{\lambda_{laser}}{\lambda_{GW}}$$

Shot Noise

- Collect photons for a time of the order of the period of GW wave $\tau \sim 1/f_{GW}$

$$N_{photons} = \frac{P_{laser}}{hc/\lambda_{laser}}\tau \sim \frac{P_{laser}}{hc/\lambda_{laser}}\frac{1}{f_{GW}}$$

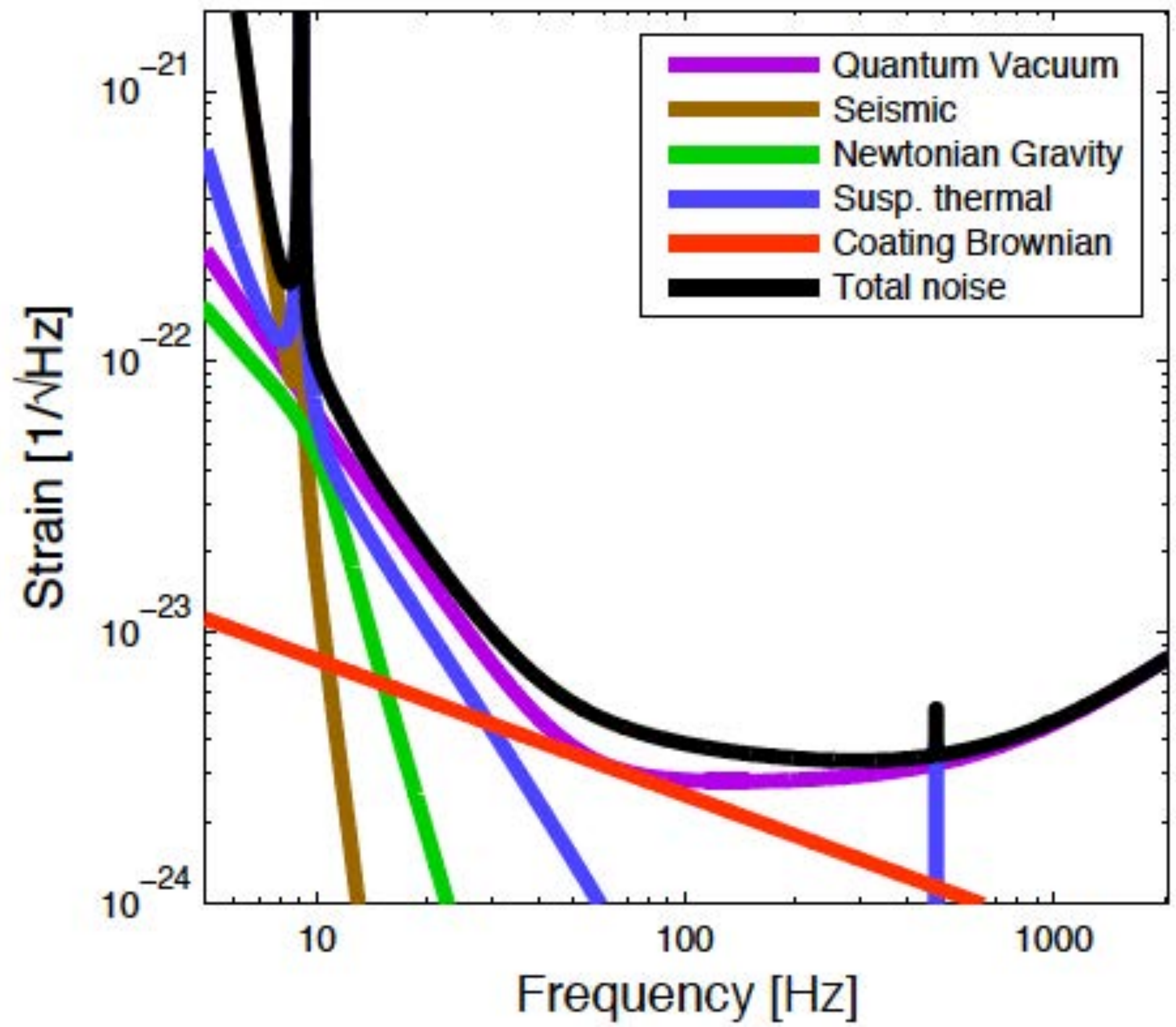
- For 1W laser with λ_{laser}=1 μm, f_{GW}=300Hz, $N_{photons}$=10^{16}

$$h \sim \frac{\Delta l}{l_{eff}} \sim \frac{N_{photons}^{-1/2}\lambda_{laser}}{\lambda_{GW}} \sim \frac{10^{-8}\times 10^{-6}\mathrm{m}}{10^{6}\mathrm{m}} = 10^{-20}$$

- By adopting high power laser (20W for O1) and power recycling, we can reach 'astrophysical sensitivity' of ~10^{-22}.

Other Noises

- Radiation pressure noise
  - Make mirrors heavier
- Suspension thermal noise/ mirror coating brownian noise
  - Increase beam size, monolithic suspension structure
- Seismic noise
  - Multi-stage suspension, underground
- Newtonian Noise
  - So far difficult to avoid.
  - Seismic and wind measurement and careful modeling



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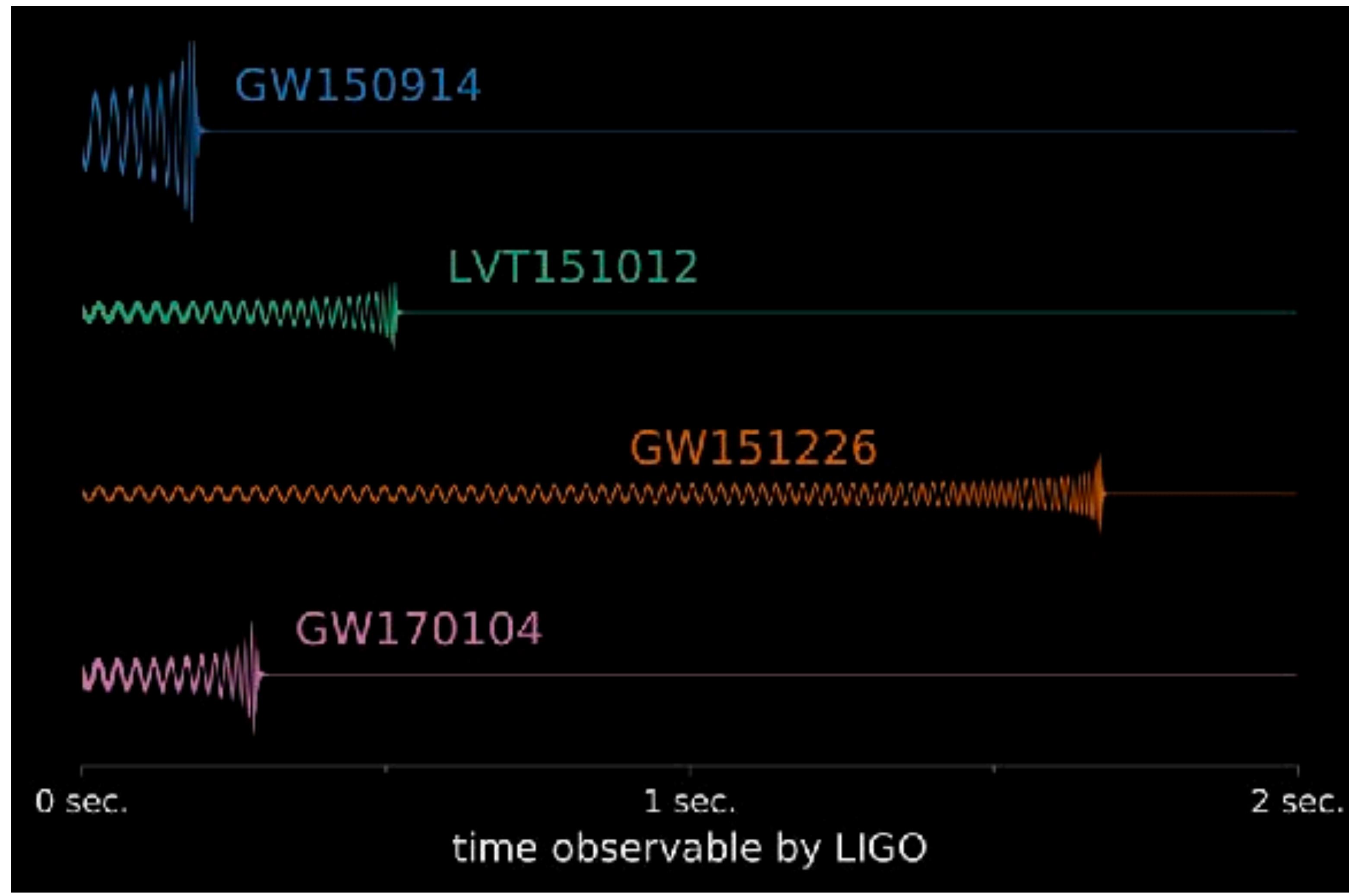
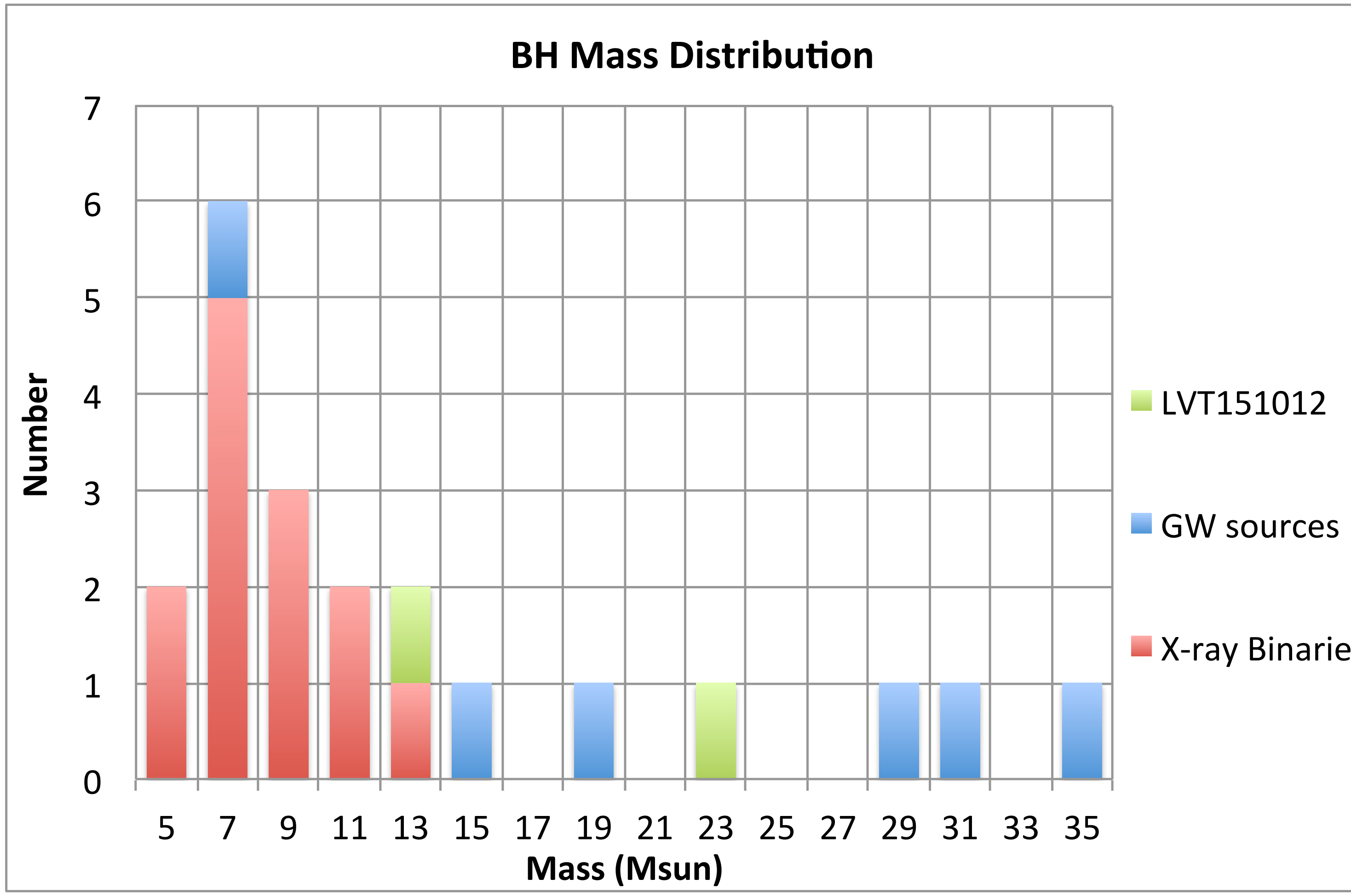
GW Events from O1/O2

- GW150914 (FAR$<6\times10^{-7}$ yr^{-1})
- LVT151012 (Candidate, FAR$\sim$0.37 yr^{-1})
- GW151226 (FAR$<6\times10^{-7}$ yr^{-1})
- GW170104 ($<5\times10^{-5}$ yr^{-1}



What made these detections interesting?

- LIGO detected gravitational waves from merger of **black hole binaries**, instead of neutron star binaries
- **Black hole mass range was quite large:** GW150914 is composed of 36 and 29 times of the mass of the Sun
  - Most of the known black holes are much less massive (~10 times of the sun)



What these results tell us?

(Abbott et al., 2016, ApJL, 828, L22; PHYSICAL REVIEW X 6, 041015 (2016) PRL 118, 221101 (2017))

- Proof of the existence of the black holes
- Existence of stellar mass black holes in binaries
- Individual masses in wide range (7-35 Msun)
- Formation of ~60 Msun BH
- BH appears to be much more frequent than previously thought
  - Current estimation of the rate 12-210 yr^{-1} Gpc^{-1}

Summary of the GW sources

- GW150914:
  - First Unambiguous detection of stellar mass black holes and a BH binary
  - Accurate measurement of black hole masses (within ~10%)
  - Higher mass of stellar mass BH than previously thought: low metallicity environment?
- GW151226:
  - Lower masses than GW150914, similar to the X-ray binary BH mass
  - Lower mass progenitor or high metallicity environment?
- GW170104
  - High mass (~50 Msun)
  - Spin may not be aligned
- Origin
  - Isolated or dynamical?

References

S. Weinberg, "Gravitation and Cosmology: Principles and applications of the general theory of relativity", 1972, John Wiley & Sons

J. B. Hartle, "Gravity, an introduction to Einstein's General Relativity". 2003, Addison Wesley

Lecture notes by K. Thorne:

- https://www.lorentz.leidenuniv.nl/lorentzchair/thorne/Thorne1.pdf
- http://www.pmaweb.caltech.edu/Courses/ph136/yr2012/ (together with R. Blandford, Chapters 25 & 27)

Recent LIGO papers

- Abbott et al. 2016, "Abbott et al. 2016, "Observation of Gravitational Waves from a Binary Black Hole Merger", 2016, PRL, 116, 061102
- Abbott et al., 2016, "ASTROPHYSICAL IMPLICATIONS OF THE BINARY BLACK HOLE MERGER GW150914" , ApJL, 828, L22
- Abbott et al. 2016, "GW150914: Implications for the Stochastic Gravitational-Wave Background from Binary Black Holes", PRL, 115, 131102
- Abbott et al. 2016, "GW150914: The Advanced LIGO Detectors in the Era of First Discoveries", 2016, PRL, 116, 131103
- Abbott et al., 2017, “GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence”, PRL, 118, 221101