Dynamics of star clusters containing stellar mass black holes: 2. Dynamical Evolution Driven by two-body Relaxation

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Stellar Systems

- **Star clusters (open, globular), galactic nuclei, galaxies, clusters of galaxies**
- **Self-gravitating**
- **Composed of mostly point masses**
- **Weak field (i.e., Newtonian dynamics)**
- **Question: What is the fate of the self-gravitating system of point masses interacting according to Newton's laws?**

Basic Parameters

- Globular Clusters
  - $N \sim 10^5$, D~70pc, age~10 Gyr (~age of the universe)
- Open Clusters
  - $N \sim 10^3$, D~20pc, age~300 Myr
- Galactic Nucleus
  - $N \sim 10^8$, D~200pc, age~10 Gyr (~age of the universe)
- Clusters of Galaxies
  - $N \sim 10^5$, D~70pc, age~10 Gyr (~age of the universe)

Strong versus Weak Encounters

- Strong encounters lead to large changes in velocity while weak encounters change velocity by only small amounts
  - $\Delta v \sim v$ for strong encounters
  - $\Delta v << v$ for weak encounters
- One strong encounters changes the orbit significantly .
- Weak encounters change the orbit slowly: accumulated effects are important

Strong Encounter Time Scale

- Condition :

$$\frac{Gm^2}{r} > \frac{1}{2}mv^2 \quad \rightarrow r_s = \frac{2Gm}{v^2} \;\; ; \quad r_s : \text{Encounter Radius}$$

- Cross section: $\sigma_s = \pi r_s^2$

- Time scale:

$$t_s = \langle n\sigma_s v\rangle^{-1}$$

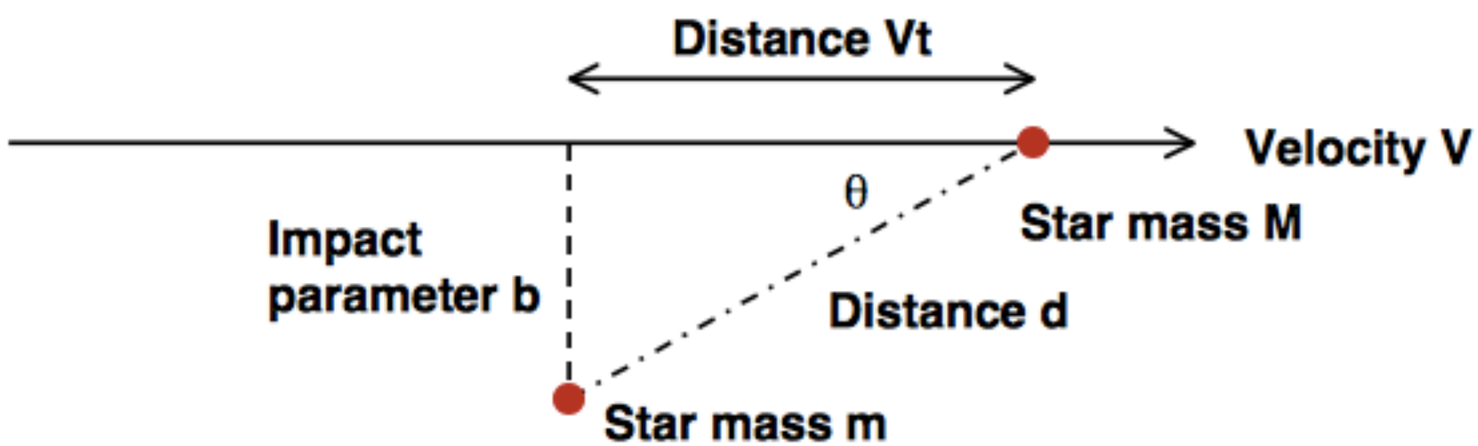
$$\approx \frac{v^3}{4\pi G^2 m^2 n}$$

$$\approx 4\times 10^{12}\left(\frac{v}{10\mathrm{km/s}}\right)^3\left(\frac{m}{\mathrm{M}_\odot}\right)^{-2}\left(\frac{n}{1\mathrm{pc}^{-3}}\right)^{-1} \;\; \mathrm{yr}$$

- Strong encounters could be quite frequent for n>10^3 pc^{-3}.

Weak Encounters

- Encounters with $b >> r_s$ would perturb the orbits



- Assuming straight line (impulse approximation) for the orbit of M, force acting on M due to m (t=0 at minimum distance)

$$F = \frac{GMm}{d^2} = \frac{GMm}{b^2 + v^2 t^2}$$

Velocity Changes

- The perpendicular component of the force

$$F_\perp = F\sin\theta = \frac{GMm}{d^2}\frac{b}{d} = \frac{GMm}{\left(b^2+v^2t^2\right)^{3/2}}$$

- Integration over time from -∞ to +∞:

$$\Delta v_\perp = \int_{-\infty}^{\infty}\frac{dv_\perp}{dt}dt = \frac{1}{M}\int_{-\infty}^{\infty}F_\perp dt = \frac{2Gm}{bv}$$

- Since $\Delta v_\perp$ has random direction, direct summation would cancel out. Instead we sum (integrate) the squared values

$$\left\langle \Delta v_\perp^2 \right\rangle = \int_{b_{min}}^{b_{max}}\left(\frac{2Gm}{bv}\right)^2 dN, \quad dN = n\times vt\times 2\pi b db$$

$$= \frac{8\pi G^2 m^2 n t}{v}\ln\left(\frac{b_{max}}{b_{min}}\right)$$

Relaxation time

- During relaxation, accumulated change of squared velocity equals the squared velocity itself.

$$t_{relax} = \frac{v^3}{8\pi G^2 m^2 n \ln \Lambda}; \quad \Lambda \equiv b_{max}/b_{min}$$

- By comparing with strong encounter time scale, we obtain

$$t_{relax} = \frac{t_s}{\ln \Lambda}$$

- For self-gravitating systems,

$$d_{max} = R_{cluster} \approx \frac{GM_{tot}}{v^2}; \quad r_s = \frac{2Gm}{v^2},$$

$$\rightarrow \Lambda = \frac{d_{max}}{d_{min}} \sim 0.5 M_{tot}/m = 0.5N, \text{ number of stars}$$

Comparison with other time scales

- Dynamical time: time for a star to make a single trip across the entire system

$$t_{dyn} \approx \frac{v}{R}$$

- If the system is in virial equilibrium

$$t_{\rm rel} \approx \frac{0.1N}{\ln N} t_{dyn}$$

$t_{\rm rel} >> t_{\rm dyn}$ if N becomes large (> 1000)

Half-mass relaxation time

- Relaxation time is locally defined.
- More precise derivation of the relaxation time can be done by taking into account the velocity distribution:

$$t_{relax} = \frac{0.34\sigma^3}{G^2 m^2 n \ln \Lambda}; \quad \sigma : 1-\mathrm{D\ velocity\ dispersion}$$

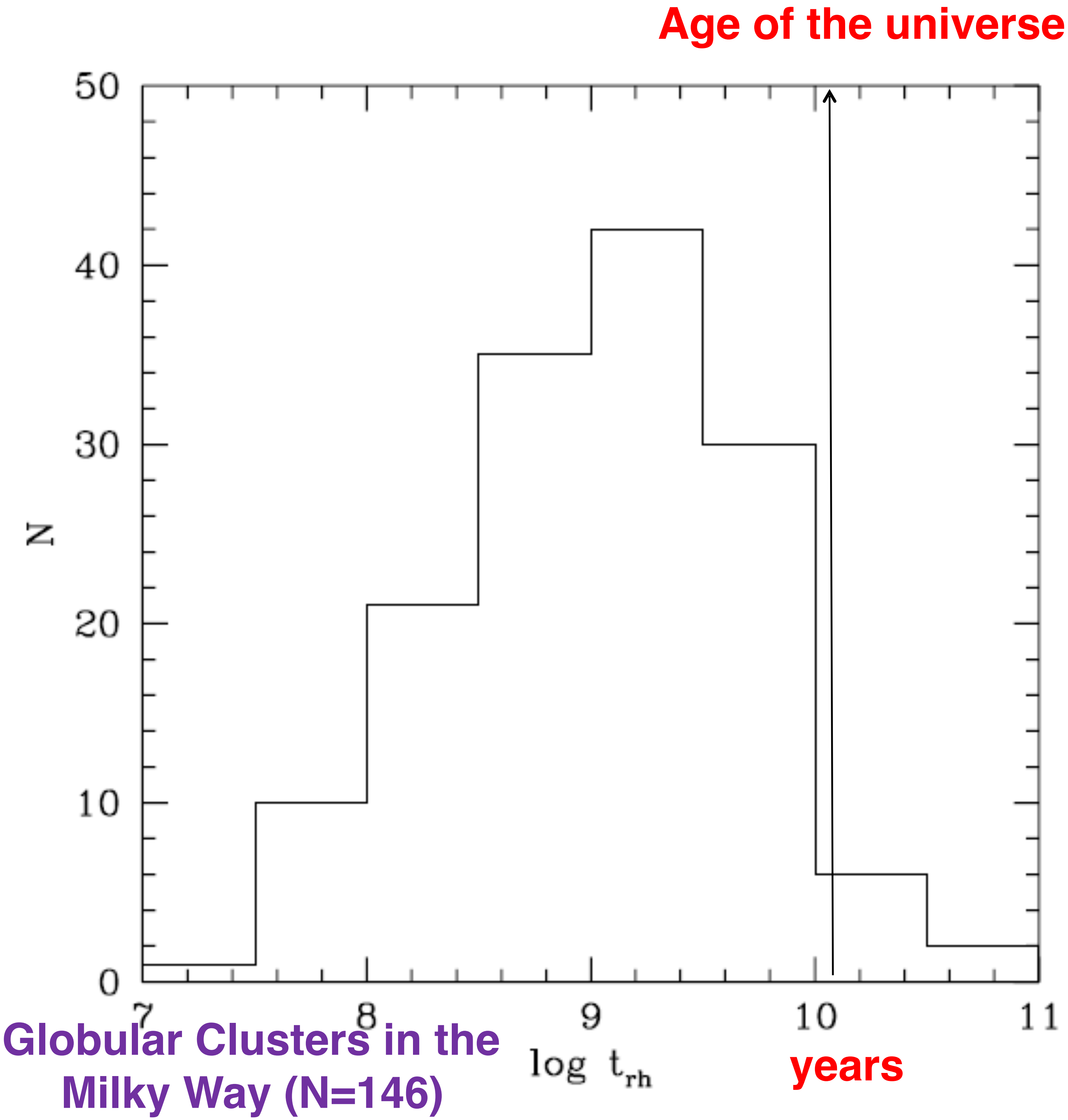
- Also half-mass relaxation time is often used as a characteristic relaxation time for the entire cluster

$$t_{rh} = 0.138 \frac{N^{1/2} r_h^{3/2}}{m^{1/2} G^{1/2} \ln \Lambda}$$

where N is the total number of stars in the cluster

Collisional or collisionless

- During $t \ll t_{rel}$, stellar orbit is determined by the smooth potential of the system: collisionless
- Otherwise, the stellar orbit will deviate from the original one: collisional
- Many globular clusters are considered to be collisional systems



Consequences of Relaxation

- Core collapse (~10 t_{rel})
  - Relaxation acts as a thermal diffusion in a system having gradient of velocity dispersion
  - Heat flow from hot to cold region drives dynamical evolution
- Mass segregation (~ t_{rel} x [low mass/high mass])
  - Stars with different mass tend to reach energy equipartition.
  - Higher mass stars lose energy to lower mass ones, and thus sink toward the center
- Evaporation (~20 t_{rel})
  - Relaxation process tends to produce Maxwellian velocity distribution
  - Stars with $v > v_{esc}$ evaporate from the cluster

Thermodynamic properties of self-gravitating systems: core collapse

- Temperature: $$k_B T = \frac{1}{2} m \sigma^2$$
- Virial Theorem:

$$E = K + V = -3Nk_B T = -\zeta \frac{GM^2}{r}$$

- Specific heat: $$\frac{\partial E}{\partial T} = -3Nk_B < 0$$

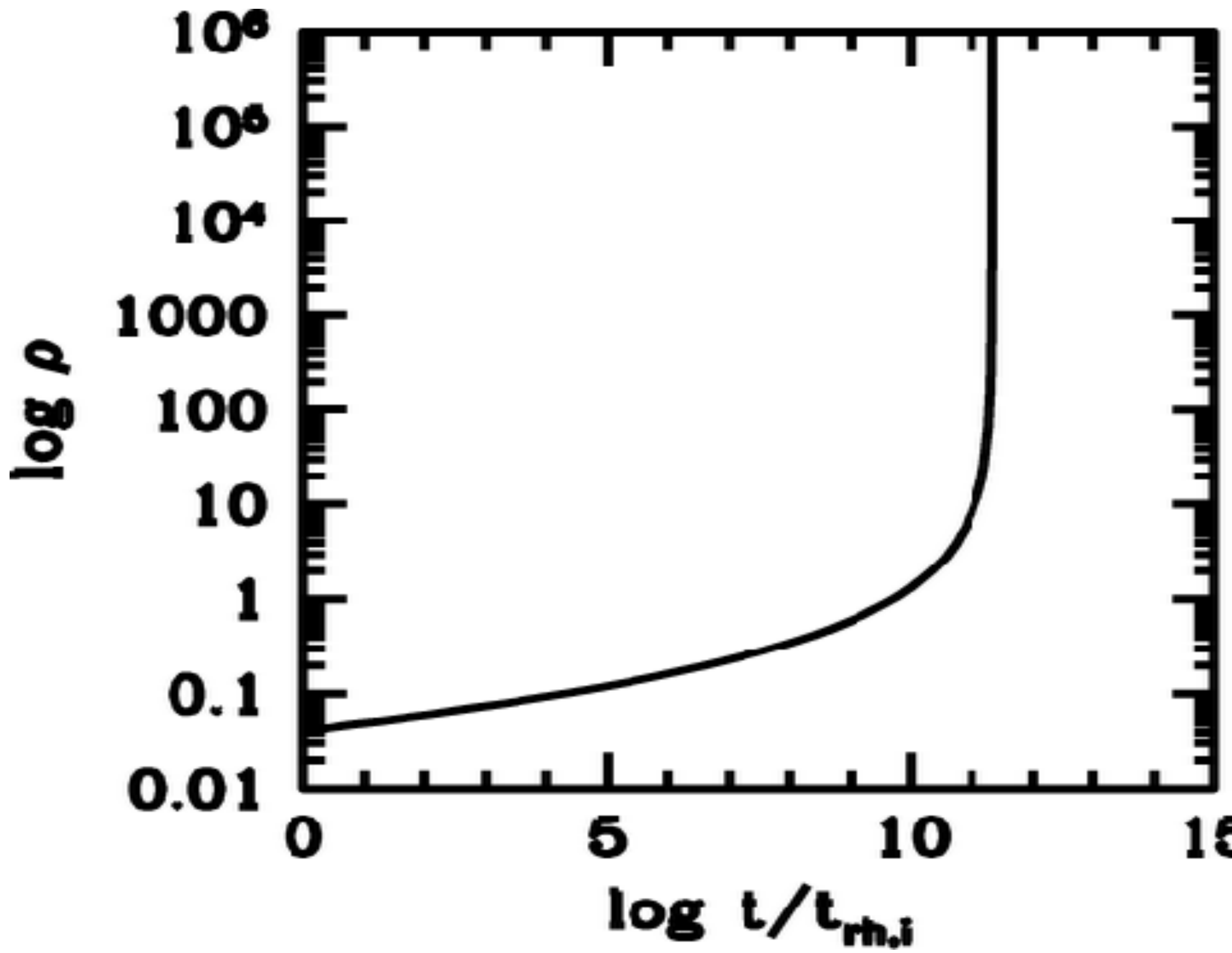
—> When energy is extracted, the system heats up!

Self-gravitating systems are thermodynamically unstable: gravothermal instability

Core collapse (or gravothermal catastrophe)

- The central density becomes infinite in finite time: gravothermal catastrophe.
- In real systems, infinite density would never happen as collisional effects become important

Based on the integration of Fokker-Planck equation



Dynamical Friction

- Consider the motion of a body of a mass *M* through a population of stars of *m*.
- Position and velocity vectors of two particles: ($\mathbf{x}_M$, $\mathbf{v}_M$), ($\mathbf{x}_m$, $\mathbf{v}_m$)
- Relative position and velocity: ($\mathbf{r}=\mathbf{x}_m-\mathbf{x}_M$, $\mathbf{V}=d\mathbf{r}/dt$)
- Equation of motion of the reduced mass:

$$\frac{mM}{m+M}\ddot{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{e}}_{\mathbf{r}}$$

- Let $\Delta\mathbf{v}_m$ and $\Delta\mathbf{v}_M$ be changes of $\mathbf{v}_m$ and $\mathbf{v}_M$ during the encounter.
- Calculate Δv_M for an encounter with impact parameter *b* and initial velocity at infinity of V_0.

Velocity changes of particle after the encounter

- Decompose $\Delta \mathbf{v}_M$ into vertical and parallel components, $\Delta \mathbf{v}_{M\perp}$ $\Delta \mathbf{v}_{M\|}$

$$\Delta \mathbf{v}_{M\perp} = \frac{2mbV_0^3}{G(M+m)}^2 \left[1 + \frac{b^2 V_0^4}{G^2(M+m)^2}\right]^{-1}$$

$$\Delta \mathbf{v}_{M\|} = \frac{2mV_0}{(M+m)} \left[1 + \frac{b^2 V_0^4}{G^2(M+m)^2}\right]^{-1}$$

- Vertical components cancel out, but parallel component accumulates and M suffers steady deceleration: dynamical friction

Collective Effects

- The rate at which M encounters stars that have velocities at impact parameters between *b* and *b+db*:

$$2\pi b db \times V_0 \times f(\mathbf{v}_m) d^3\mathbf{v}_M$$

  where $f(\mathbf{v}_m)$ is the phase-space number density of stars with mass *m*.

- Net rate of change of $\mathbf{v}_M$ due to these encounters:

$$\left.\frac{d\mathbf{v}_M}{dt}\right|_{v_m} = \mathbf{V_0} f(v_m) d^3\mathbf{v}_m \int_0^{b_{max}} \frac{2mV_0}{M+m}\left[1+\frac{b^2V_0^4}{G^2(M+m)^2}\right]^{-1}$$

$$= 2\pi \ln(1+\Lambda^2) G^2 m(M+m) f(\mathbf{v}_m) d^3\mathbf{v}_m \frac{(\mathbf{v}_m - \mathbf{v}_M)}{|\mathbf{v}_m - \mathbf{v}_M|^3}$$

where $\Lambda \equiv \dfrac{b_{max}V_0^2}{G(M+m)}$

Integration over all velocities

- The previous equation for dv/dt has the same form for the calculation of the gravitational force at the position vector v_M that is generated by the mass density

$$\rho(\mathbf{v}_M) \equiv 4\pi \ln \Lambda G m(M+m) f(\mathbf{v}_m)$$

- Therefore dv/dt is equivalent to G/v_M^2 times mass in $v_m < v_M$.

$$\frac{d\mathbf{v}_M}{dt} = -16\pi^2 \ln \Lambda G^2 m(M+m) \frac{\int_0^{v_M} f(v_m) v_m^2 dv_m}{v_M^3} \mathbf{v}_M$$

- For small v_M, $f(v_m) \rightarrow f(0)$, and therefore

$$\frac{d\mathbf{v}_M}{dt} \approx -\frac{16\pi^2}{3} \ln \Lambda G^2 m(M+m) f(0) \mathbf{v}_M$$

- For large v_M, the integral converges to definite limit

$$\frac{d\mathbf{v_M}}{dt} = -4\pi G^2 M m n \ln \Lambda \frac{\mathbf{v}_M}{v_M^3}$$

General Result and Friction Time

- For arbitrary vM and Maxwellian velocity distribution for background stars

$$\frac{d\mathbf{v_M}}{dt} = -\frac{4\pi G^2 M m n \ln\Lambda}{v_M^3}\left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}}\mathrm{e}^{-X^2}\right]\mathbf{v}_M; \quad X \equiv v_M/(\sqrt{2}\sigma)$$

- Thus dynamical time scale becomes

$$t_{fric} = \frac{v_m}{|dv_m/dt|} = 2\frac{m}{M}\left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}}\mathrm{e}^{-X^2}\right]t_{rel}$$

- Generally the dynamical evolution is accelerated by the presence of the dynamical friction

Evaporation of Stars

- The stars evaporate from cluster when $v > v_e$ where v_e=-2**Φ** is the escape velocity.
- The mean square of the escape velocity is

$$< v_e^2 >= \frac{\int \rho(x) v_e^2 d^3x}{M} = -2\frac{\int \rho(x)\Phi(x) d^3x}{M} = -4\frac{V}{M}$$

where V is the potential energy, and M is the total mass.

- Since $V=-2K = -M<v^2>$

  $<v_e^2> = 4<v^2>$.

Tidal Radius

- Size of a cluster is limited by the tidal field of the Galaxy
- In a frame of the cluster which is rotating at speed **Ω**, equation of motion becomes

$$\ddot{\mathbf{x}} = -\nabla\Phi - 2\mathbf{\Omega} \times \dot{\mathbf{x}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x})$$

$$= -\nabla\Phi_{eff} - 2\mathbf{\Omega} \times \dot{\mathbf{x}}$$

where

$$\Phi_{eff} \equiv \Phi - \frac{1}{2}|\mathbf{\Omega} \times \mathbf{x}|^2$$

- There exist a surface where

$$\nabla\Phi_{eff} = 0$$

- Tidal radius: $r_t = \left(\frac{m_{cl}}{2M_G}\right)^{1/3} R_G$

Evaporation Rate

- Suppose the stellar system reaches velocity distribution *f(v)* in relaxation time (t_{rh}). Then the evaporation rate per relaxation time becomes

$$\xi_e = -\frac{t_{rh}}{M}\frac{dM}{dt} = \frac{\int_{v_e}^{\infty} f(v) d^3v}{\int_0^{\infty} f(v) d^3v}.$$

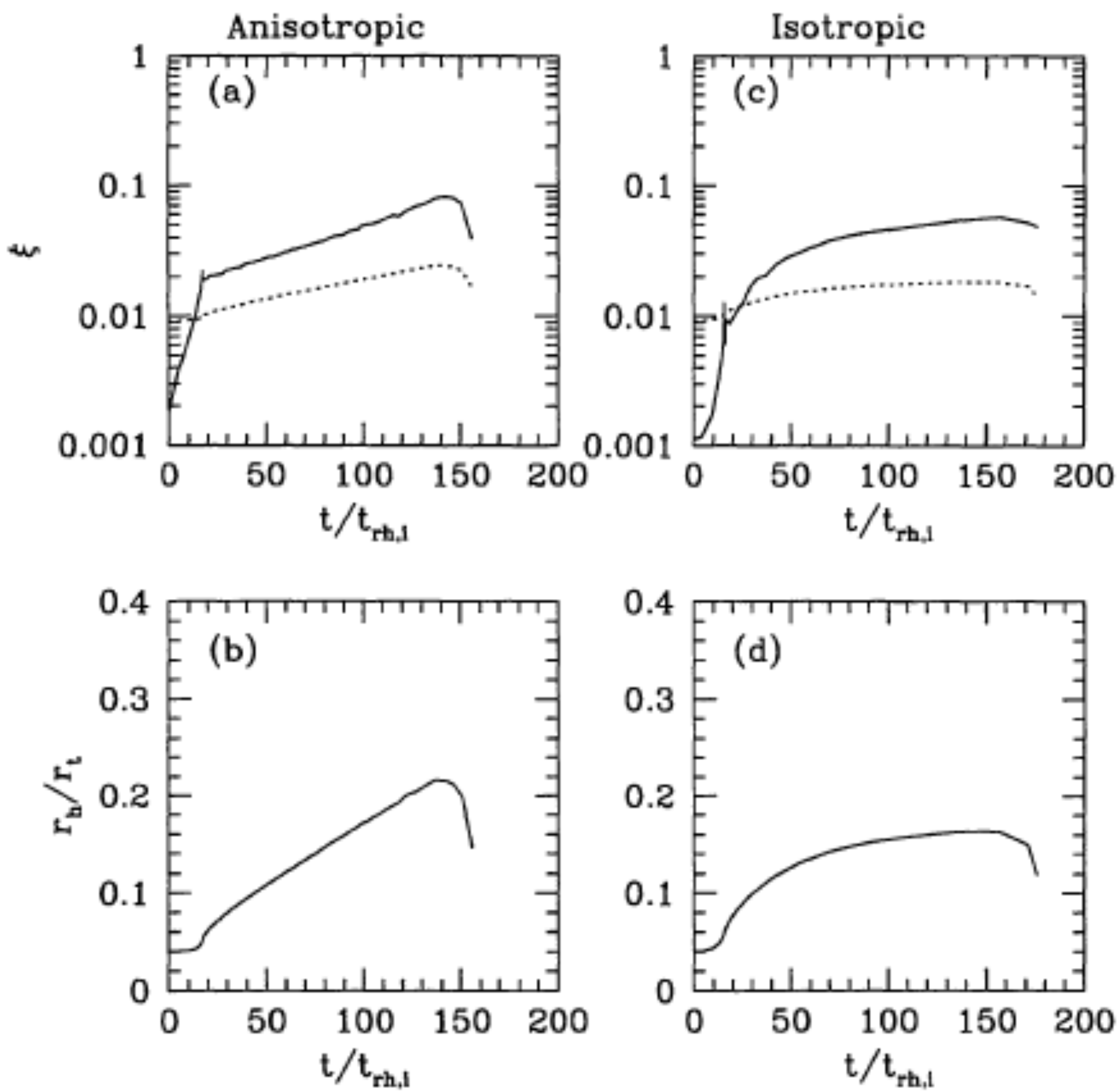
- If f(v) is a Maxwellian, ξ_e =0.0073.

- For tidally bound systems, escape velocity is reduced (Takahashi, Lee & Inagaki 1997)

$$< v_e^2 > = 4(1-\lambda) < v^2 >$$

$$\lambda \approx \frac{5r_h}{4r_t}$$

Takahashi, Lee & Inagaki 1997



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