Dynamics of star clusters containing stellar mass black holes: 3. Methods

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Methods of Modelling of Dynamical Evolution

- Direct integration of equation of motion (N-body method):
  - very time consuming (proportional to N^3)
  - possible for systems with $N<10^5$, whereas sometimes we are interested in $N> 10^6$ systems

- Statistical methods: Fokker-Planck equation, Gas dynamical approach, Monte Carlo method: Fast, but many restrictions

  spherical symmetry

  simple velocity dispersion tensor

  more physics other than dynamical relaxation

- Both statistical and direct N-body methods can be used in a complimentary way

Direct Integration of Equations of Motion

- Newton's Equation of Motion

$$\frac{d^2\mathbf{r}_i}{dt^2} = -\sum_{j\neq i}^{N} \frac{Gm_j}{(r_i - r_j)^3}(\mathbf{r}_i - \mathbf{r}_j)$$

  - *N* equations of *N*-1 terms in summation → *N(N-1)/2* equations
  - For large N, number of operations increases with N^2.
  - Furthermore, number of time steps for the meaningful evolution ∝N ➔ Number of operations for the completion of the study ∝ N^3.
  - Direct integration is difficult and time consuming for N > 10^5

N-body Codes

- Many versions exists
- N-body codes implement various techniques to deal with complex situations
  - Multiple time steps
  - Regularization
  - Neighbor schemes
- Special purpose hardwares have been developed
  - GRAPE (Gravity Pipe)
- Recent versions take advantages of GPU and parallel computers
- The most popular version N-body6 (more recently N-body7) can be downloaded from S. Aarseth's homepage

  https://www.ast.cam.ac.uk/~sverre/web/pages/nbody.htm

  http://sverre.com/index.php?page=intro
- It is not difficult to run these codes. The most important part is to set up appropriate initial conditions and to choose right options.
- Running N-body code requires long computing time, typically a few days to a few months

Statistical Treatment: Phase space distribution function

- For systems with large N, statistical treatment may be possible (i.e., like gaseous systems)

- For such cases, we may use ‘phase space distribution function’ and compute the evolution of the distribution function

$f(x,v)d^3xd^3v$: number of particles in phase space volume $(x,v)\sim(x+d^3x,\ v+d^3v)$

Boltzmann's Equation

- A conservation equation in phase space in the absence of 'collision'

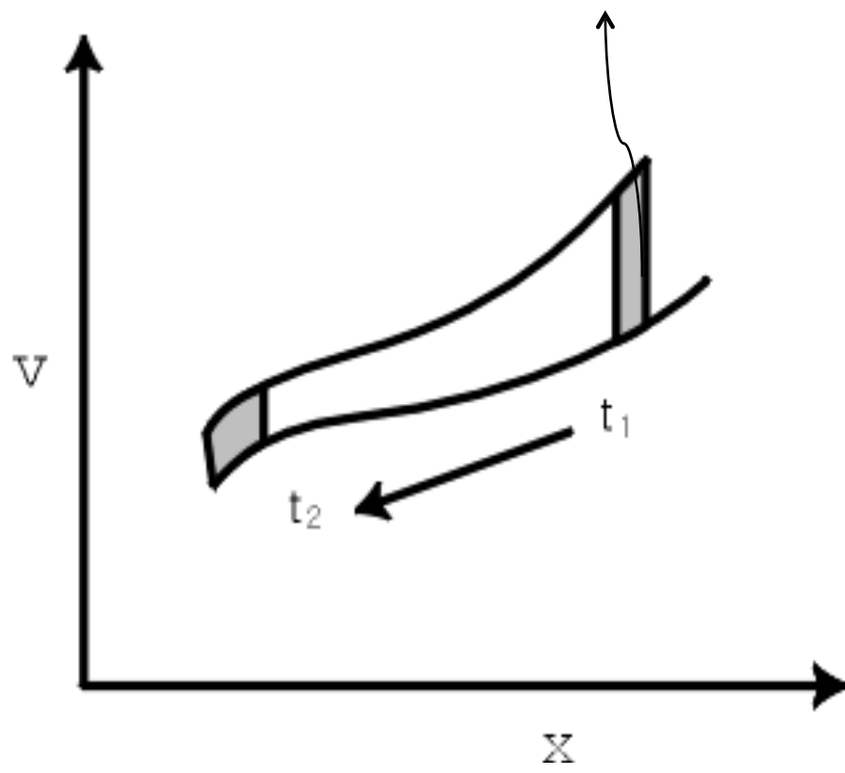
$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(f\dot{x_i}) + \frac{\partial}{\partial v_i}(f\dot{v_i})$$

collisional

- It is similar to the continuity equation of the fluid:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

- The collisions may lead to jump of a particle in phase space. If there is no collision, the collisional term becomes 0.: collision less Boltzmann equation



Integrals of Motion

- Depending on the geometric shape of the potential, there exists a number of conserved quantities which depends only on (x,v)

- These are called ‘integrals (or constants)’ of motion that isolate the nature of orbits

- The integrals themselves satisfy the collisionless Boltzmann’s equation:

$$\frac{dI}{dt} = \sum_{i=1}^{3} \left(\dot{x}_i \frac{\partial I}{\partial x_i} + \dot{v}_i \frac{\partial I}{\partial v_i} \right) = \sum_{i=1}^{3} \left(v_i \frac{\partial I}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial I}{\partial v_i} \right) = 0.$$

Jeans Theorems

- Jeans Theorem: The distribution function of collisionless systems depends on phase space variables only through the integrals of motion, i.e., $f=f(I_1, I_2,\ldots)$ since

$$\frac{Df}{Dt} = \sum_j \frac{DI_j}{Dt}\frac{\partial f}{\partial I_j} = 0.$$

- The distribution can evolve over time longer than relaxation time --> Fokker-Planck description

Integrals of Motion for Spherical and Axisymmetric Systems

- Spherical system: Energy (E) and angular momentum (J)
- Spherical system with isotropic velocity: E
- Rotating Axisymmetric systems: E, J along the axis of rotation (J_z), and I_3 (unknown third integral).

Calculation of Collisional Term

- The changes in f due to two-body scattering is described by the transition probability function $\Psi(w,\Delta w)$, which is the probability that a star with w=(p,q) is scattered to the volume space $d(\Delta^6 w)$ around $\mathbf{w}+\Delta\mathbf{w}$ during Δt.

$$\left.\frac{\partial f(w)}{\partial t}\right|_{-} = -f(w)\int d^6(\Delta w)\Psi(w,\Delta w)$$

$$\frac{\partial f(w)}{\partial t} = \int d^6(\Delta w)\Psi(w-\Delta w,\Delta w)f(w-\Delta w)$$

$$\left(\frac{df}{dt}\right)_{coll} = \Gamma[f] = \left.\frac{\partial f}{\partial t}\right|_{+} + \left.\frac{\partial f}{\partial t}\right|_{-}$$

Fokker-Planck Approximation

- The Fokker-Planck assumes that only small scattering dominates the evolution. In that case, we can expand Ψ up to second order in Δw.

$$\Psi(w-\Delta w,\Delta w)f(w-\Delta w)=\Psi(w,\Delta w)f(w)$$

$$-\Sigma\Delta w_i\frac{\partial}{\partial w_i}[\Psi(w,\Delta w)f(w)]$$

$$+\frac{1}{2}\Sigma\Sigma\Delta w_i\Delta w_j\frac{\partial^2}{\partial w_i\partial w_j}[\Psi(w,\Delta w)f(w)]$$

More Approximations

- Local approximation: the encounter makes changes only in velocity → $\Delta \boldsymbol{x}=0$.

- Orbit averaging: since the orbit is nearly conserved with small angle encounters, one can compute $\Delta \boldsymbol{v}$ over one single orbit, and then convert it to the rate of change

- Then convert $\Delta \boldsymbol{v}$ into $\Delta \boldsymbol{I}$**'s.**

- The resulting equation can be expressed as a diffusion equation in integral space:

Fokker-Planck Equation

- Jeans Theorem may be applied to the collision less systems in steady state.

- Effects of Stellar Encounters
- The collisional effects are assumed to be due to small angle scattering events. Since it takes long time to evolve, Jeans theorem can be assumed to hold each time. Then we may write

$$f = f(I_1, I_2, I_3, ...t)$$

- Fokker-Planck equation describes the evolution of distribution function in time

- Under small angle encounter assumption, the evolution equation becomes a set of diffusion equations in integrals of motion

Spherical Systems

- Distribution function for spherical systems depends on E and J, with E=1/2 v^2 + $\Phi(r)$, J=rv_t

- Let $N(E,J)$: total number of stars within the interval dE and dJ

$$N(E,J)dEdJ = 2\int_{r_p}^{r_a} f(E,J)4\pi r^2 dr v^2 dv \times 2\pi \sin\theta d\theta$$

- θ: angle between r and v. Since

$$dJ = r dv_t = rv\cos\theta d\theta = rv_r d\theta$$

$$\sin\theta d\theta = \frac{v_t dJ}{rvv_r} = \frac{JdJ}{r^2 vv_r}$$

$$N(E,J) = 16\pi^2 f(E,J)J\int_{r_p}^{r_a}\frac{dr}{dv_r}$$

$$= 8\pi^2 f(E,J)JP(E,J)$$

Orbit averaged Fokker-Planck equation as a diffusion equation in (E,J) space

- Formally orbit averaged Fokker-Planck equation can be written as

$$\left(\frac{\partial N}{\partial t}\right) = -\frac{\partial}{\partial E}\left[N < \Delta E >_{orb}\right] - \frac{\partial}{\partial J}\left[N < \Delta J >_{orb}\right] + \frac{1}{2}\frac{\partial^2}{\partial E^2}\left[N < (\Delta E)^2 >_{orb}\right]$$
$$= \frac{\partial^2}{\partial E \partial J}\left[N < \Delta E \Delta J >_{orb}\right] + \frac{1}{2}\frac{\partial^2}{\partial J^2}\left[N < (\Delta J)^2 >_{orb}\right]$$

where orbit averaged diffusion coefficients can be obtained by, for example,

$$< \Delta E >_{orb} = \frac{2}{P(E,J)} \int_{r_p}^{r_a} < \Delta E > \frac{dr}{v_r}$$

Adiabatic Invariants

- Physical quantities that remain nearly constant under slowly varying potential.
- In spherical potential, radial action $Q(E,J)$, in addition to J, is an adiabatic invariant.

$$Q(E,J) \equiv 2\int_{r_p}^{r_a} v_r dr$$

- In isotropic case, we can define an adiabatically invariant function that depends only on energy by integrating

$$q(E) = \frac{1}{2}\int_0^{J_{max}} Q(E,J) J dJ$$

Fokker-Planck equation in for spherical systems with isotropic velocity dispersion

- Isotropic: velocity dispersions are identical along any directions
- In that case, the distribution function depends only on E.
- Fokker-Planck equation becomes particularly simple

$$\frac{\partial N}{\partial t} = -\frac{\partial F_E}{\partial E}, \qquad F_E = -D_{EE}\frac{\partial f}{\partial E} - D_E f .$$

$$D_{EE} = 16\pi^3 \Gamma q(E) \int_0^E dE' f(E') + 16\pi^3 \Gamma \int_E^{\phi(0)} dE' q(E') f(E') ,$$

$$D_E = -16\pi^3 \Gamma \int_E^{\phi(0)} dE' p(E') f(E') ,$$

$$\Gamma = 4\pi G^2 m^2 \ln \Lambda,$$

Auxiliary functions

$$q(E) = \int_0^{J_c^2(E)} d(J^2) Q(E, J)$$

$$= \frac{4}{3} \int_0^{\phi^{-1}(E)} dr\, r^2 v^3 \,,$$

Poisson's equation

$$\phi(r) = \frac{4\pi G}{r} \int_0^r dr' r'^2 \rho(r')$$

$$+ 4\pi G \int_r^\infty dr' r' \rho(r') \,,$$

Fokker-Planck equation in self-gravitating systems

- As the system evolves, the gravitational potential also changes

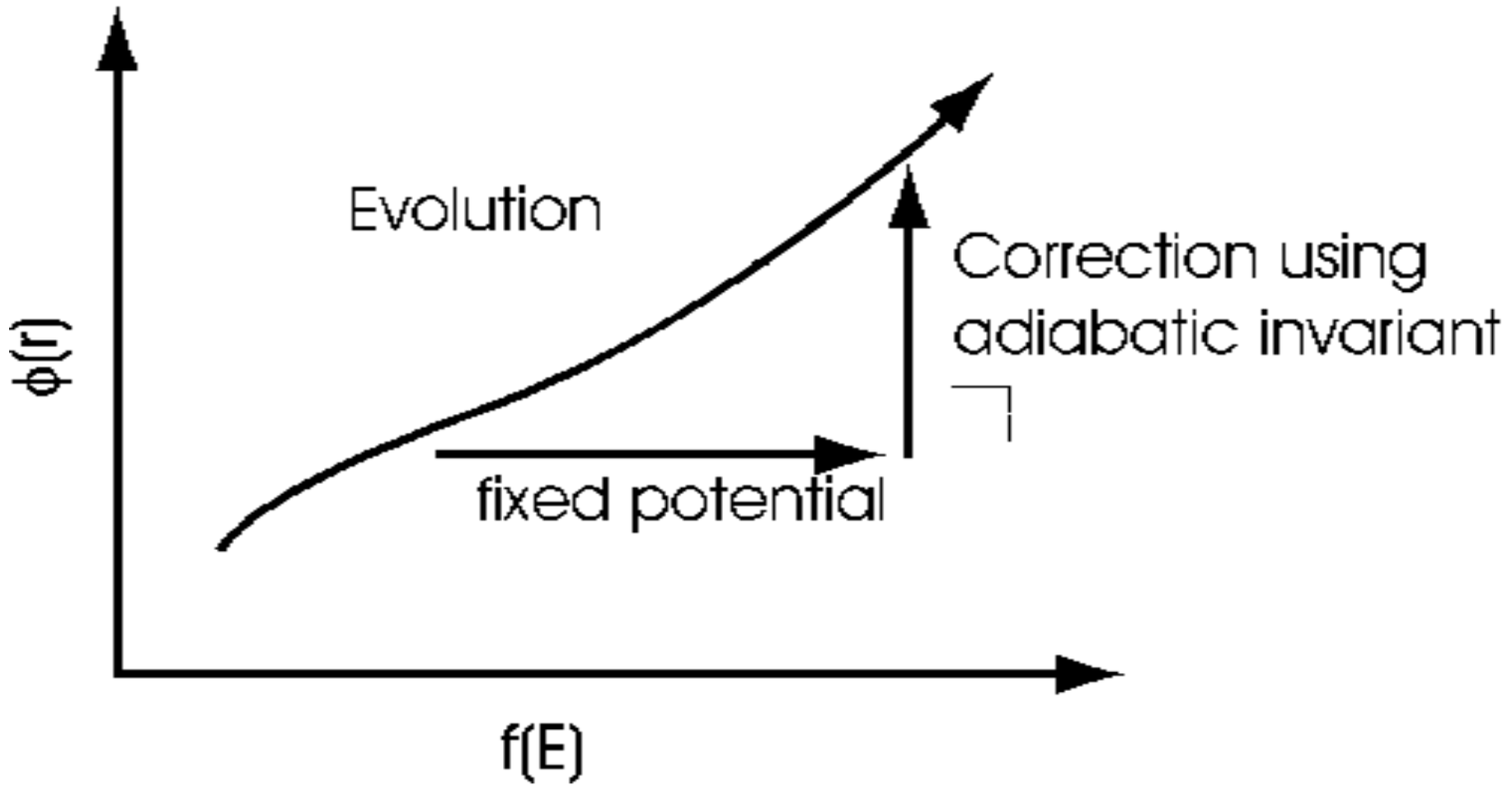
→ integrals (I) may experience additional changes

- The Fokker-Planck equation assumes that the potential is fixed: we need some modification of the simple Fokker-Planck equation

- Self-consistent distribution function and gravitational potential must be computed

—> utilize the adiabatic invariants as a constraint equation

A Procedure to obtain self-consistent potential



Application of Fokker-Planck equation to stellar dynamics

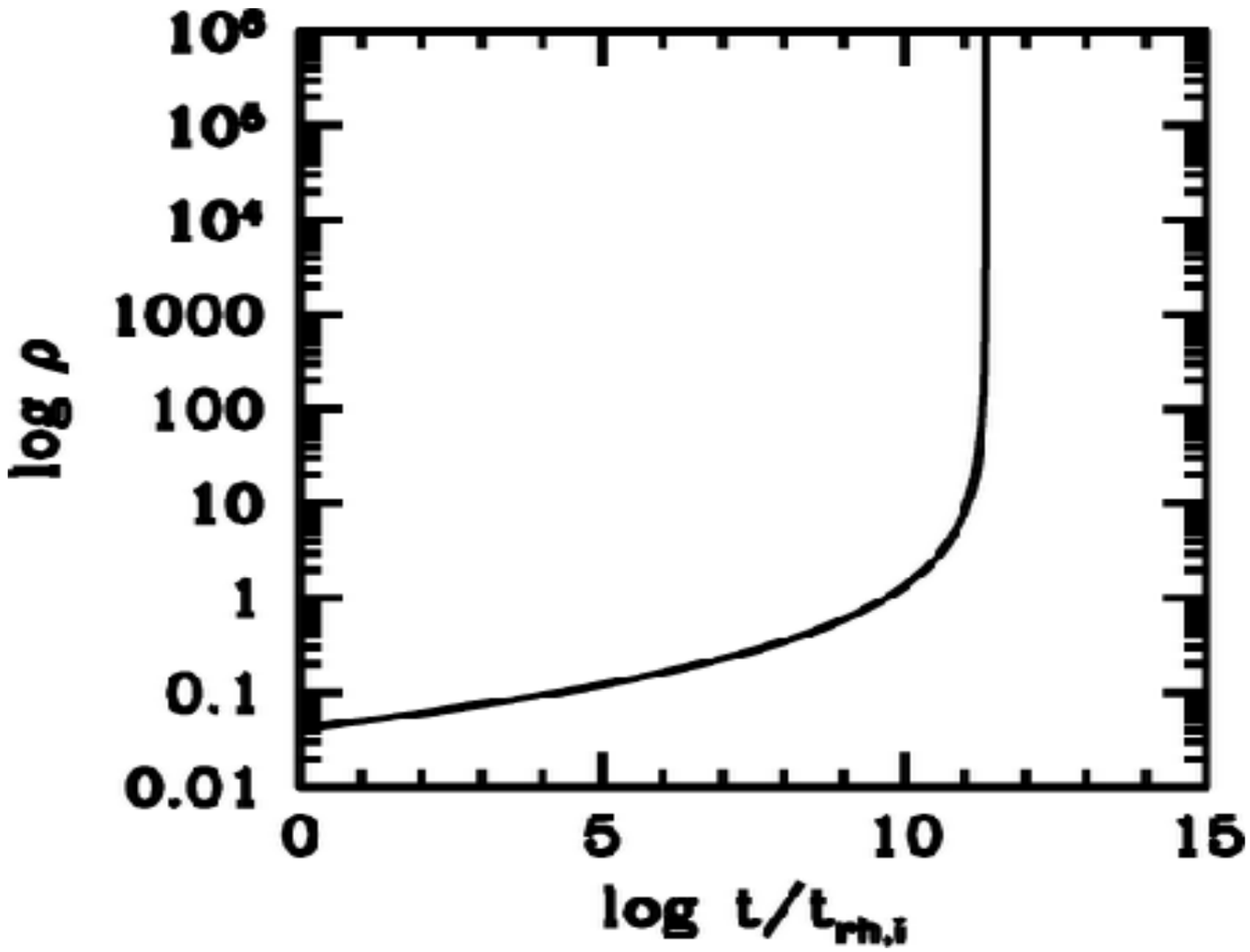
- Formalism of FP equation was done by plasma physicists in 1970s

- FP equation was introduced to stellar dynamics by Kulsrud & Cohn (1978), & Cohn (1979, 1980)

- Most of the earlier calculations assume only one integral: Energy (E)

Example: Core collapse

- Self-gravitation systems undergo gravothermal catastrophe via two-body relaxation
- The central part becomes very dense
- Since core collapse takes place indefinitely, no evolution is possible beyond it.
- The time scale for this is ~10 t_{rh} .
- The evolution after the core-collapse is possible if there is a heating source.
  - Supernova explosion
  - Binary stars



Toward more realistic models: 1. Multi-mass models

- Clusters are composed of many different populations with different individual masses
- Extension to the multi-mass system is straightforward: We need to integrate the following F-P equations for a discrete mass distribution denoted by index i with mass m_i

$$\frac{\partial N_i(E)}{\partial t} = -\frac{\partial F_i(E)}{\partial E}; \quad F_i(E) = -D_{EE}\frac{\partial f_i(E)}{\partial E} - D_E f_i(E)$$

$$D_{EE} = 16\pi^3\Gamma\sum_i m_i^2\left[q(E)\int_0^E dEf_i(E) + \int_E^{\phi(0)} dEq(E)f_i(E)\right]$$

$$D_E = -16\pi^3\Gamma\sum_i m_i\int_0^E dEf_i(E)$$

Toward more realistic models: 2. physical processes

- In real clusters, various processes occur in addition to gravitational encounters, e.g., stellar evolution, formation/destruction of binaries, inelastic (superelastic) encounters , external field (constant or time varying)

➔ Modification of FP equation with source and sink terms (Statler, Lee , Ostriker, Goodman…. 1986 ~ 2000)

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial E}\left(-D_E f - D_{EE}\frac{\partial f}{\partial E} - H_E f - H_{EE}\frac{\partial f}{\partial E} \right) - Lf + A$$

Usual two-body relaxation term

Loss Term

Source Term

Toward more realistic models: 3. Velocity Anisotropy and Rotation

- Velocity anisotropy naturally arises during the dynamical evolution, mostly radial anisotropy in the outer parts
- Original F-P code was developed for f(E,J) by Cohn and collaborators (1978, 1979, but the numerical integration of the anisotropic code was poor.
- Substantial improvement was made by Koji Takahashi (1995, 1996) and successfully applied to clusters in a series of papers with collaborators such as H. M. Lee, Sungsoo Kim, Portgies Zwart, etc. It is still actively used by a Korean group.
- Extension to the rotating clusters was initiated by C. Einsel and R. Spurzem (Heidelberg) successfully applied to realistic clusters by our group at SNU.
  - For rotating clusters, the unknown third integral is ignored

Application Example: Dynamical Friction (Bae & Lee in preparation)

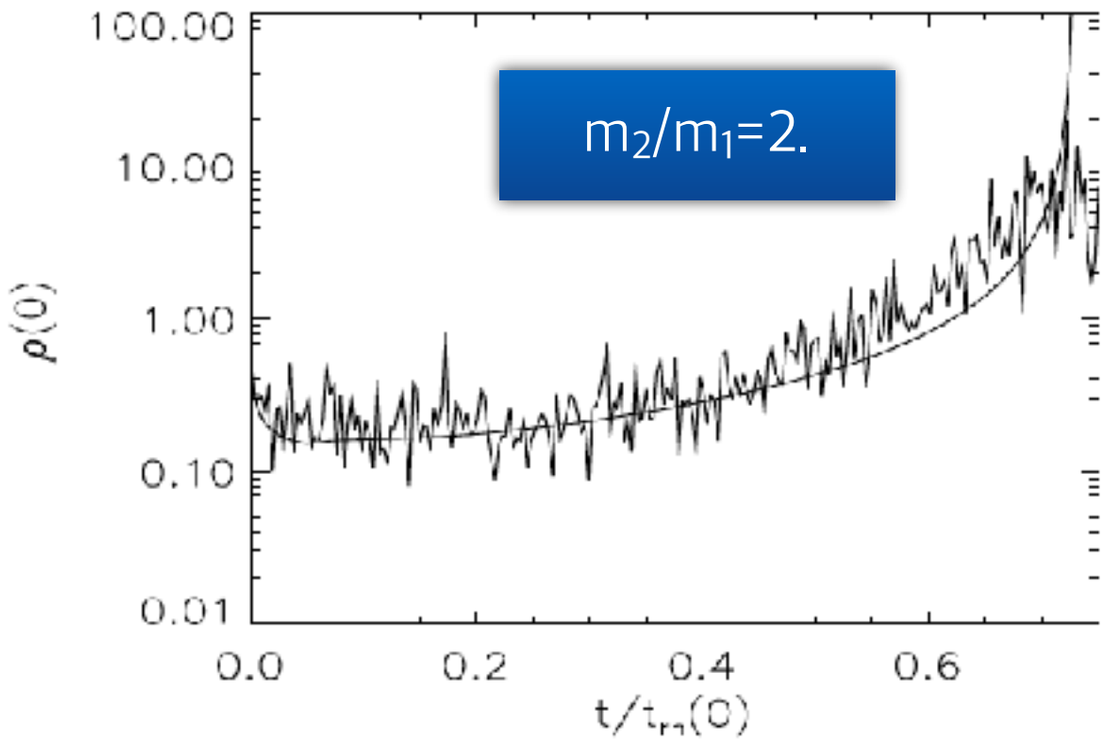
- Dynamical friction depends on mass ratio of two components: m_2/m_1.

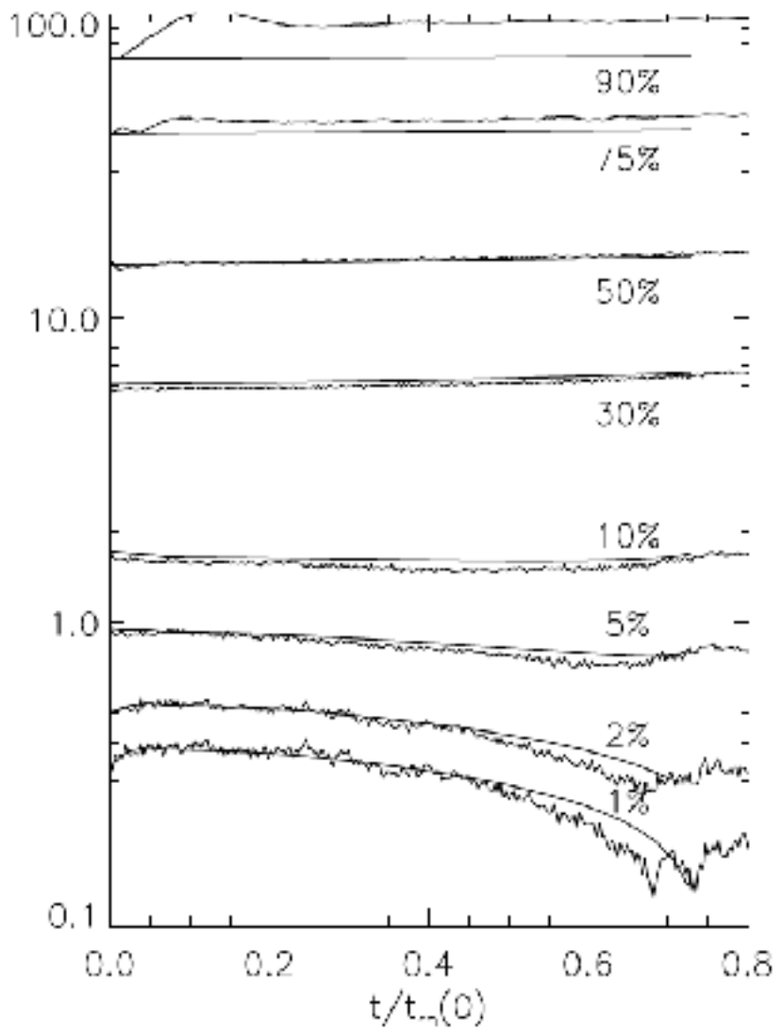
- Suppose that the galaxy is composed of two components: $m_2 >> m_1$. Massive component can be star clusters and low mass component may be single stars or dark matter particles (no difference as long as $m_2 >> m_1$).

- The global evolution of the spherical systems can be well studied with Fokker-Planck. N-body simulation requires uninhibitedly large number of particles.

Test of Fokker-Planck against N-body

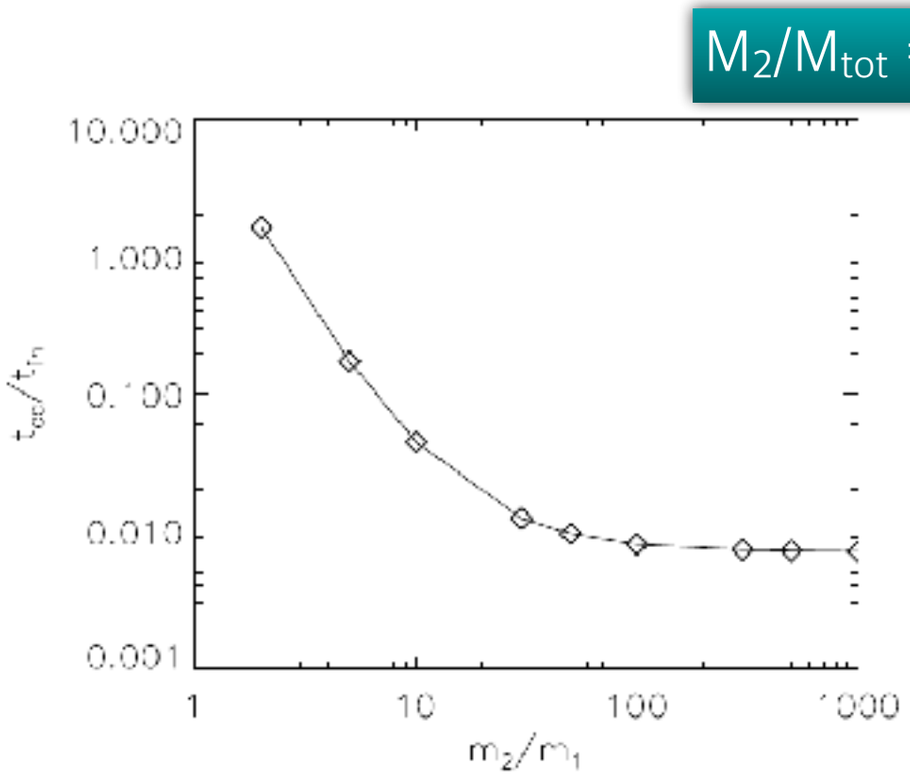
- F-P is known to work very well for initial models with flat core (i.e., King models, Plummer model, etc.)
- We also applied F-P equation for cuspy initial models for and found good agreement with NBODY6

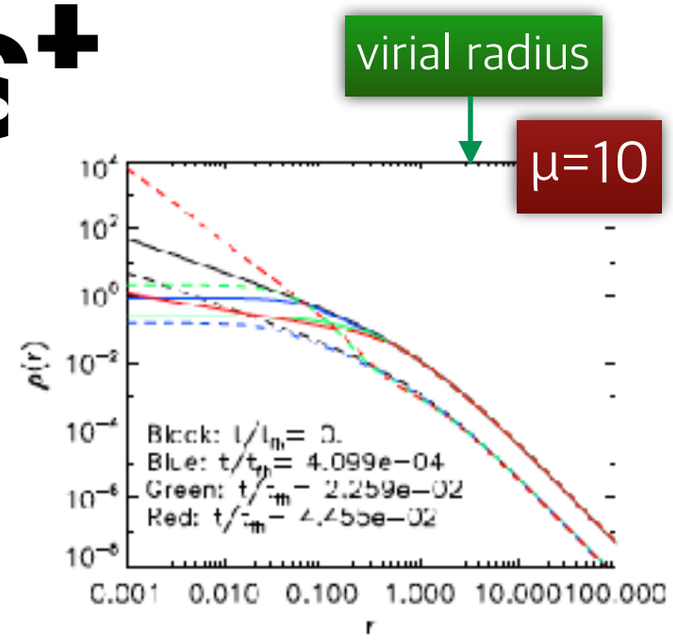


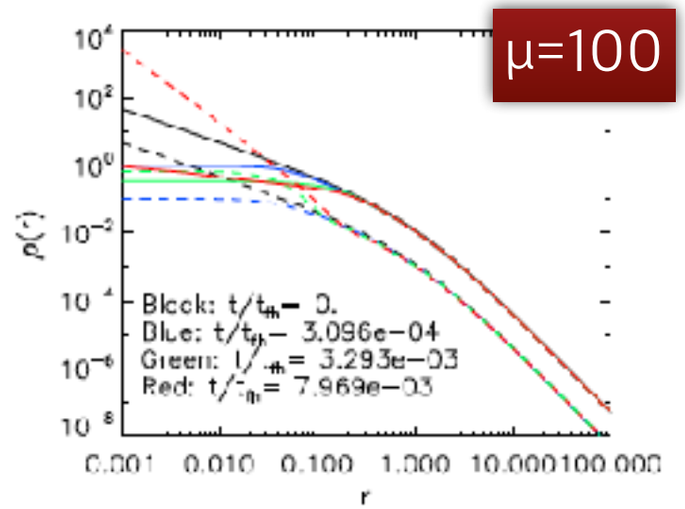


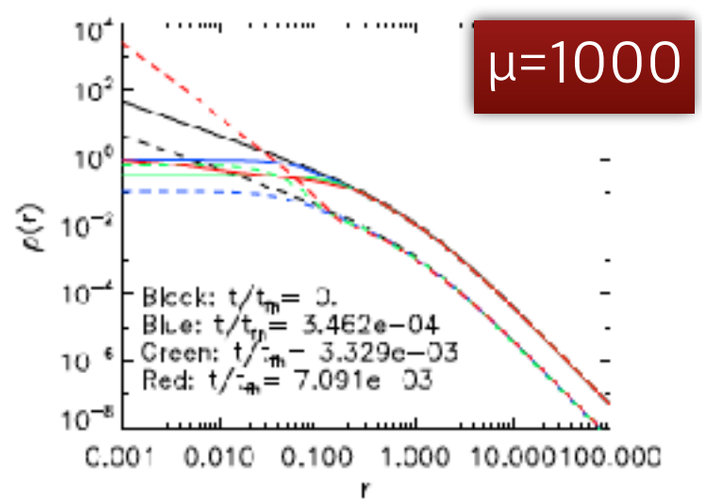
Convergence Test

- In two-component models, The evolution depends on μ=m_2/m_1. ($m_2>m_1$)
- One cannot have arbitrarily large value for μ.
- The evolution, measured by ($t_{cc}/t_{fh}\rightarrow 7.1 \times 10^{-3}$) becomes independent of μ for large μ>1000. The distinct core develops in short time!

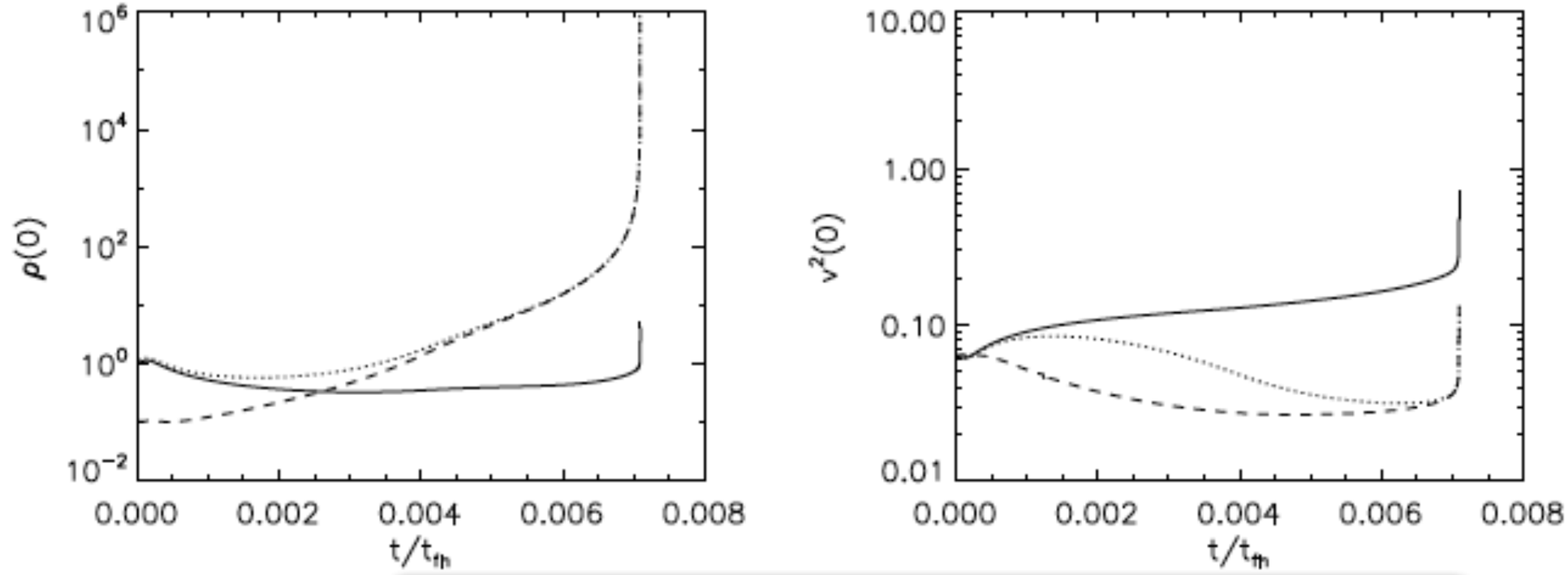






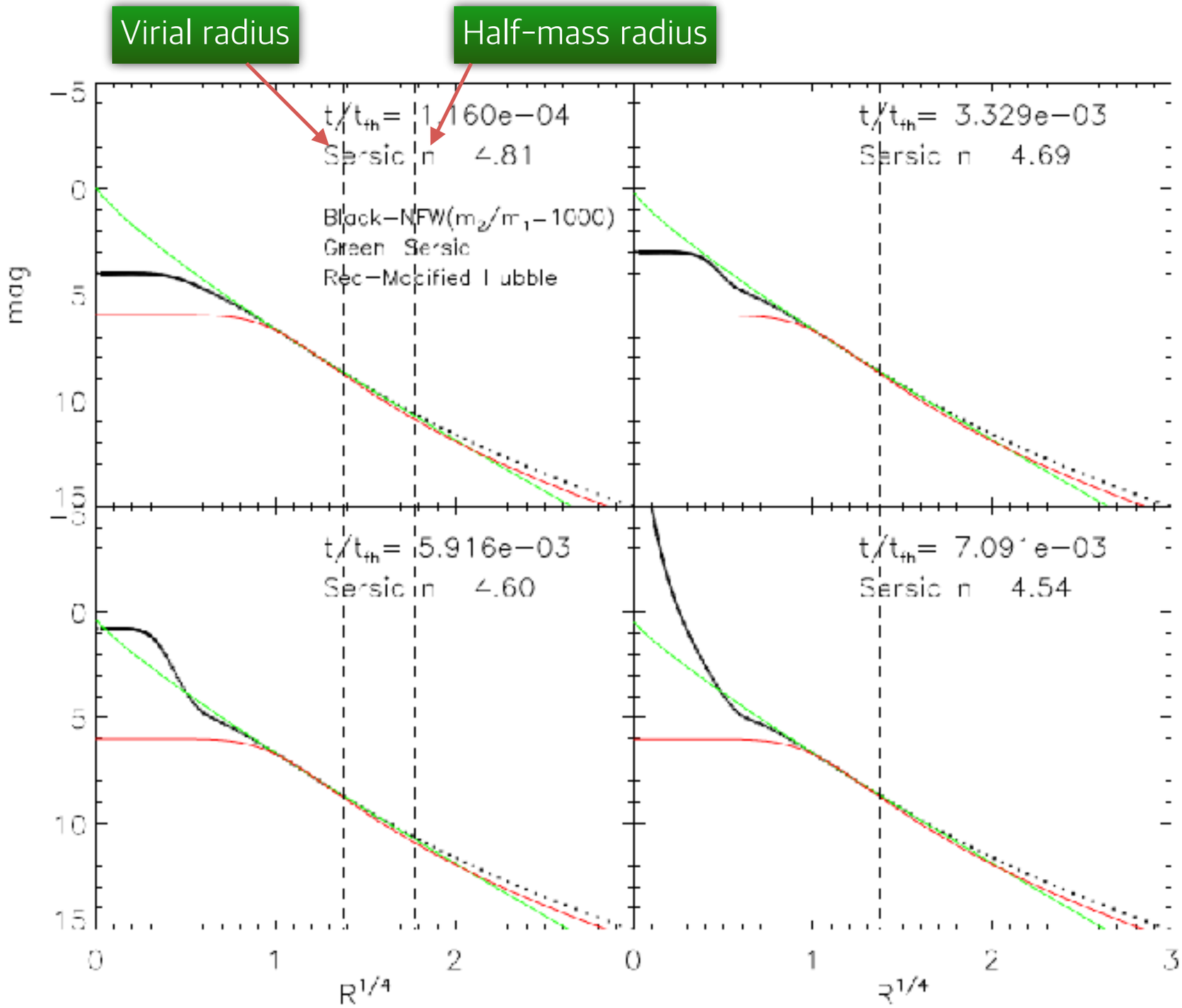


Evolution of the central density and velocity dispersion



solid: lower mass component (stars or dark matter)
broken: higher mass component (e.g., Cluster)
dotted: total
m1/m2=1000

Evolution of Density Profiles

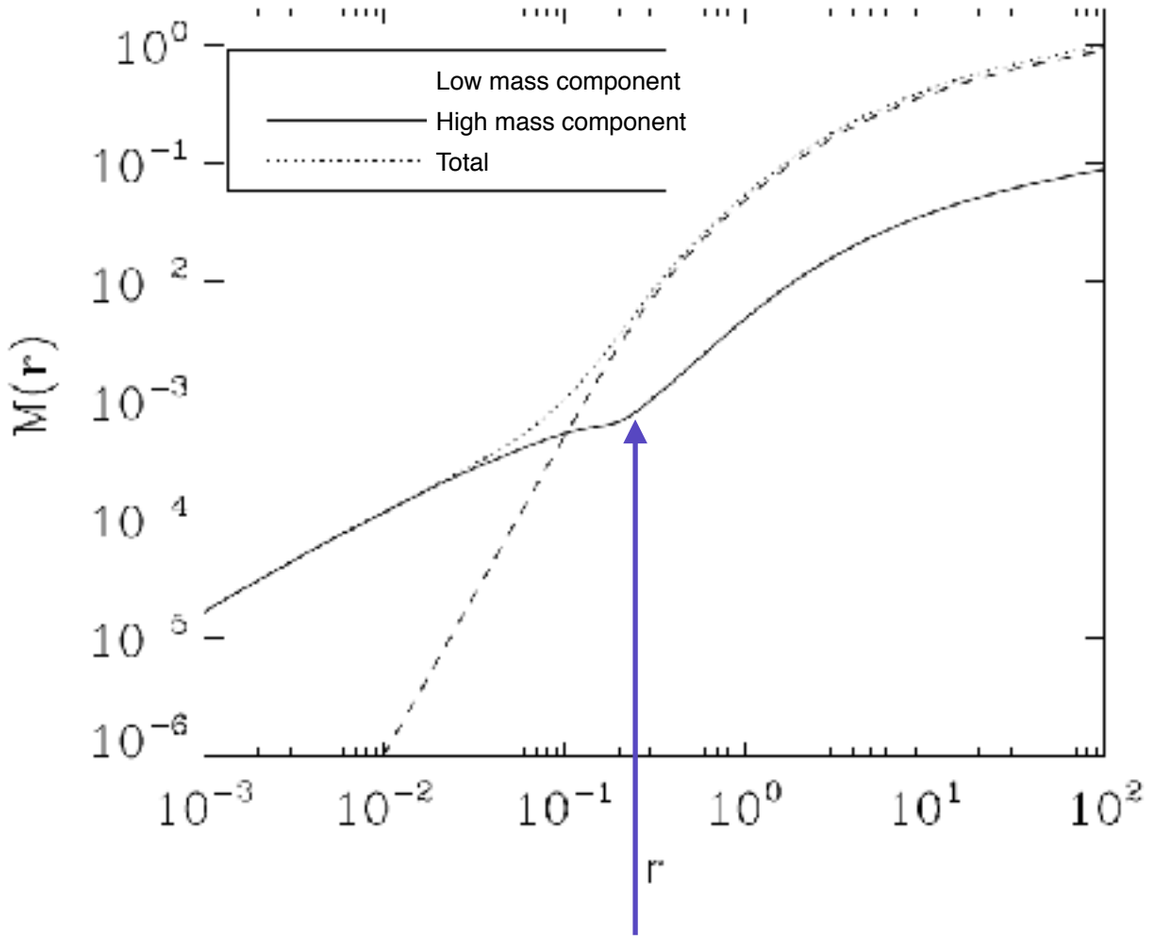


Implications

- Formation of distinctive core composed of massive component through the core collapse.

- Mass in this central concentration typically becomes ~ a few 10^{-4} of the total mass and a few 10^{-3} of total mass in m_2.

- Note that the mass of the SMBH is a few times 10^{-3} of the bulge mass.



List of microphysics

- Mass function and mass segregation
- Primordial Binaries
- Binary formation
- Binary-single interactions
- External tidal field (static or time varying)
- Stellar Evolution
- Mergers
- Evaporation of stars

Subject of the final lecture