Turbulence and dynamos for astrophysicists

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United Kingdom
International Boundary
District Boundary
National Capital
Major Cities
Railroad
Road
Rivers
Unnamed districts and counties have same name as their seats
0 50 100 KM
0 50 100 Miles
Norwegian Sea
SHETLAND ISLANDS
Mainland
North Sea
Lerwick
NORTH ATLANTIC OCEAN
61°
60°
1°30'W
ORKNEY ISLANDS
Mainland
Kirkwall
ORKNEY ISLANDS AREA
Thurso
Stornoway
Lewis with Harris
WESTERN ISLES AREA
North Uist
South Uist
Skye
HEBRIDES
HIGHLAND
Inverness
Loch Ness
Fort William
Aberdeen
GRAMPIAN
SCOTLAND
TAYSIDE
Dundee
Perth
Saint Andrews
Cupar
FIFE
CENTRAL
Stirling
Kirkcaldy
Mull
Isley
Glasgow
LOTHIAN
Edinburgh
BORDERS
Tweed
STRATHCLYDE
Newtown Saint Boswells
Ayr
Arran
DUMFRIES AND GALLOWAY
Dumfries
Stranraer
North Channel
U.K.
NORTHUMBERLAND
Newcastle upon Tyne
TYNE AND WEAR
Carlisle
CUMBRIA
Durham
Middlesbrough
CLEVELAND
Northallerton
Scarborough
NORTH YORKSHIRE
Isle of Man (U.K.)
Douglas
Barrow-in-Furness
York
Kingston upon Hull
HUMBERSIDE
Grimsby
Blackpool
LANCASHIRE
Preston
Wakefield
Barnsley
Liverpool
MERSEYSIDE
Manchester
Holyhead
Anglesey
Caernarfon
GWYNEDD
Mold
CLWYD
Chester
Matlock
Lincoln
Nottingham
Derby
Stafford
Shrewsbury
Leicester
King's Lynn
NORFOLK
Norwich
ENGLAND
Wolverhampton
West Bromwich
Peterborough
Aberystwyth
St.George's Channel
WALES
Warwick
Cambridge
SUFFOLK
Ipswich
Worcester
Llandrindod Wells
Northampton
Bedford
Fishguard
DYFED
Carmarthen
POWYS
Gloucester
Oxford
Aylesbury
Hertford
ESSEX
Chelmsford
GWENT
Thames
Swansea
Newport
London
WEST GLAMORGAN
MID GLAMORGAN
Bristol Channel
SOUTH GLAMORGAN
Cardiff
Bristol
Trowbridge
Reading
Kingston
Maidstone
Dover
KENT
Calais
Strait of Dover
Taunton
SOMERSET
Winchester
HAMPSHIRE
WEST SUSSEX
EAST SUSSEX
Lewes
Hastings
Chichester
DEVON
Exeter
Southampton
DORSET
Bournemouth
Newport
ISLE OF WIGHT
Dorchester
Plymouth
Torbay
CORNWALL
Truro
Penzance
ISLES OF SCILLY
English Channel
Alderney (U.K.)
FRANCE
North Sea
NORTH ATLANTIC OCEAN
Irish Sea
Londonderry
NORTHERN IRELAND
Belfast
Sligo
Dundalk
Westport
IRELAND
Galway
Dublin
Shannon
Ballycastle
Coleraine
MOYLE
Ballymoney
Limavady
North Channel
Londonderry
Strabane
Ballymena
Larne
Magherafelt
Antrim
Newtownabbey
Carrickfergus
Bangor
Cookstown
NORTH DOWN
Omagh
Belfast
Newtownards
LISBURN
CASTLEREAGH
CRAIGAVON
Dungannon
Hillsborough
ARDS
Portadown
DOWN
Enniskillen
Downpatrick
FERMANAGH
Armagh
Banbridge
NEWRY AND MOURNE
Newry
NORTHERN IRELAND
12°
9°
6°
3°
0°
57°
54°
51°

Format

- Tuesday 20, Wednesday 21, Thursday 22 September
- Lectures at 10:00–10:45, 11:00–11.45 and 12:00-12:45
- Lunch break at 12:45–13:30
- Office hour at 13:30–14:30 (Room 1.04)

The audience are strongly encouraged to ask questions at any time during the lectures and take advantage of the office hour.

Outline

1. Introduction
2. Elements of random functions
3. Phenomenology of fluid turbulence
4. Interstellar turbulence
5. Magnetohydrodynamic turbulence
6. Dynamos

Further reading

A. HYDRODYNAMIC TURBULENCE

M. Van Dyke, *An Album of Fluid Motion*. Parabolic Press, Stanford, 1982
U. Frisch, *Turbulence. The Legacy of A. N. Kolmogorov*. Cambridge Univ. Press, 1995
H. Tennekes & J. L. Lumley, *A First Course in Turbulence*. MIT Press, Cambridge, MA, 1972
J. Jiménez, Turbulence. In: *Perspectives in Fluid Dynamics*. Eds G. K. Batchelor, H. K. Moffatt & M. G. Worster. Cambridge Univ. Press, 2000
A. S. Monin & A. M. Yaglom, *Statistical Fluid Mechanics*. Vols 1 & 2. Ed. J. Lumley. MIT Press, 1971 & 1975 (Dover, 2007)
S. Panchev, *Random Functions and Turbulence*. Pergamon Press, Oxford, 1971

B. MAGNETOHYDRODYNAMIC TURBULENCE

D. Biskamp, *Magnetohydrodynamic Turbulence*. Cambridge Univ. Press, 2003

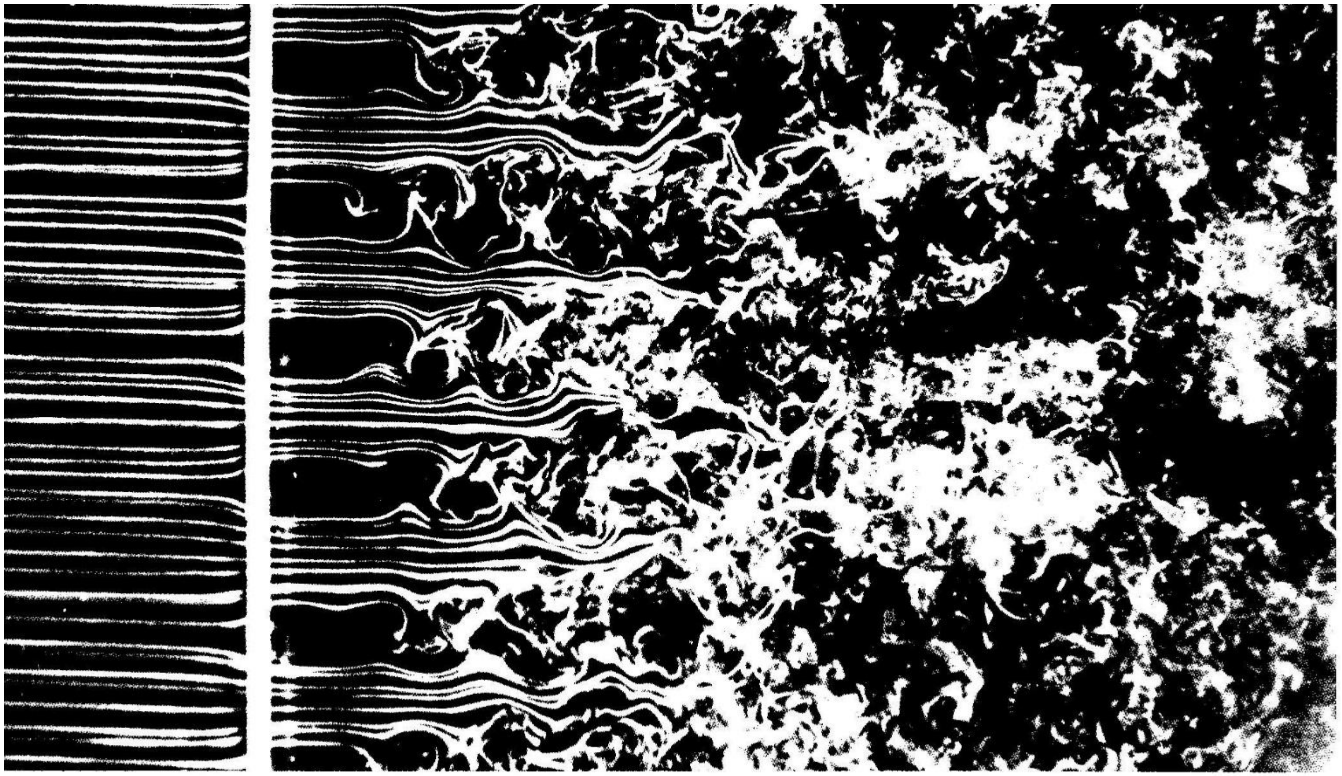
C. ASTROPHYSICAL TURBULENCE

M.-M. Mac Low & R. S. Klessen, Control of star formation by supersonic turbulence. *Rev. Mod. Phys*., 76, 125–194, 2004 (astro-ph/030193)
B. G. Elmegreen & J. Scalo, Interstellar turbulence I: Observations and processes. *Ann. Rev. Astron. Astrophys*., 2004 (astro-ph/0404451)
J. Scalo & B. G. Elmegreen, Interstellar turbulence II: Implications and effects. *Ann. Rev. Astron. Astrophys*., 2004 (astro-ph/0404452)

1

Introduction

Flows in nature: tendency to become disorderly (*turbulent)*



Turbulence behind a grid, 1 inch mesh size, the Reynolds number is 1500. The instability of shear layers leads to turbulence downstream (Fig. 152, van Dyke 1982).

Highly recommended: https://youtu.be/1_oyqLOqwnI

Figure 1: Turbulence patterns revealed by the condensation wake of the Horns Rev 1 offshore windfarm. Photographer Christian Steiness, (Credit: Vattenfall) . http://ict-aeolus.eu/about.html

https://youtu.be/NplrDarMDF8

Turbulent flows are

- highly disorganized and yet contain structures on all scales,
- velocity, pressure, etc., appear unpredictable in detail,
- and yet are reproducible in statistical sense (average values, standard deviations, etc.).

Turbulence is

a) a random flow of a liquid or gas,

b) where energy is transferred from large to small scales,

c) and dissipates there.

Each of a)–c) is an essential feature of turbulence

Turbulence requires a continuous supply of energy from

- instabilities of a laminar flow (e.g., shear instability, magneto-rotational instability in accretion discs);
- buoyancy, convection, etc.;
- external forces, e.g., supernova explosions in the ISM;
- … …

Significance of turbulence

- Augments molecular transport and causes mixing within the gas or fluid.
- Transfers energy from large scales to smaller scales where it dissipates into heat leading to enhanced viscosity, heat transfer, turbulent diffusion.
- Generates coherent structures (flow structures, large-scale magnetic fields via dynamo action).

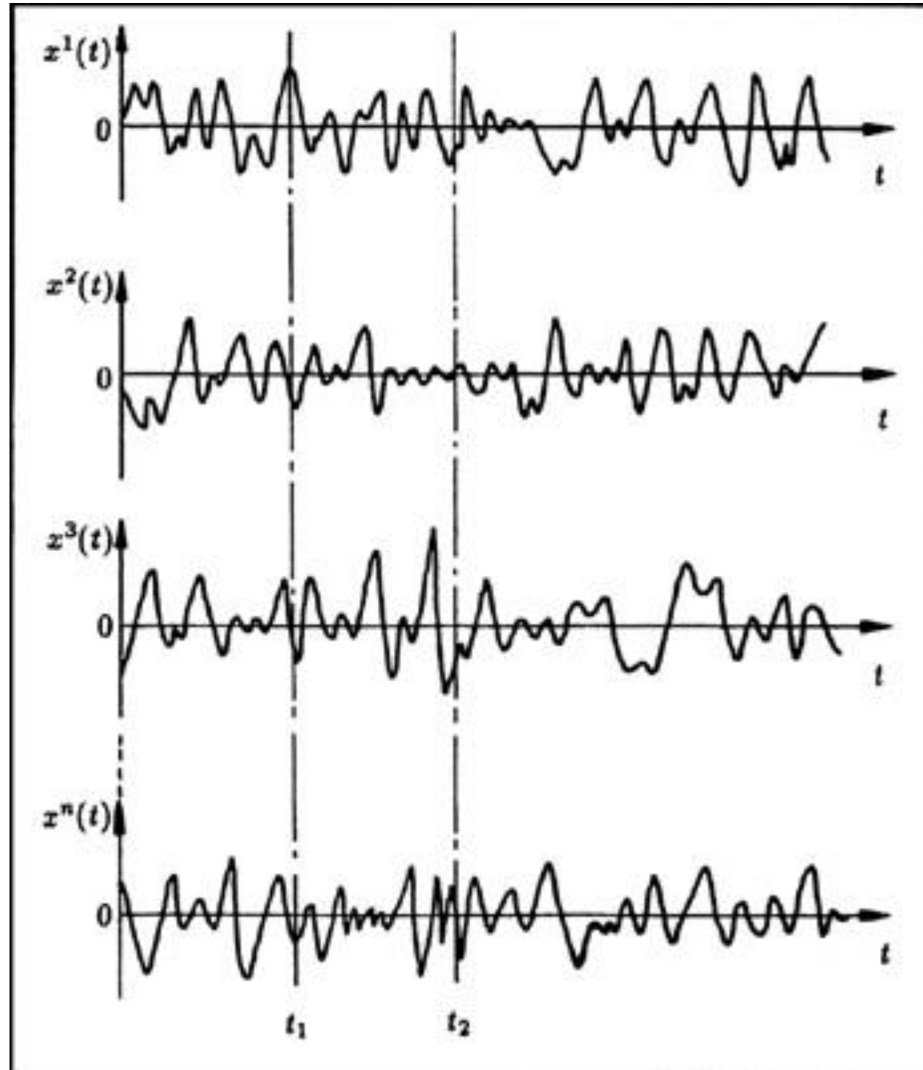
2

Elements of random functions

Turbulent flows are random

$\Rightarrow$ velocity $\vec{v}$, pressure p, magnetic field $\vec{B}$, density ρ, etc. are random functions of position $\vec{x}$ and time t.

$A(x)$ is called a *random function* of the variable x if $A(x)$ is a random variable for any fixed value x.



Realizations of a random function of time, $x(t)$

2.1. Ensemble, volume and time averaging

- Ensemble averaging: averaging over different realizations of the random function.
- Volume/time averaging: averaging of a single realization of a random function over a region in space or time interval.

Ensemble averages appear in theory but are very difficult to measure or compute as they require a very large number of realizations to converge (often, millions of realizations).

ergodic $= \varepsilon\rho\gamma o\nu$ (work) $+ \, o\delta o\zeta$ (path)

Ergodic random functions:

statistical properties obtained by averaging a set of its realizations (ensemble averages) are, with unit probability, equal to those obtained by averaging a single realization for a sufficiently long interval of time (time averages) or a sufficiently large region (volume averages).

We shall only consider ergodic random functions:

$$\underbrace{\langle A \rangle}_{\text{Ensemble average}} = \underbrace{\frac{1}{V} \iiint_V A(\vec{x}, t)\, \mathrm{d}^3 x}_{\text{Volume average}} = \underbrace{\frac{1}{T} \int_0^T A(\vec{x}, t)\, \mathrm{d}t}_{\text{Time average}}$$

2.2. Reynolds rules of averaging

For any random variables u and v, a constant c and any useful averaging procedure:

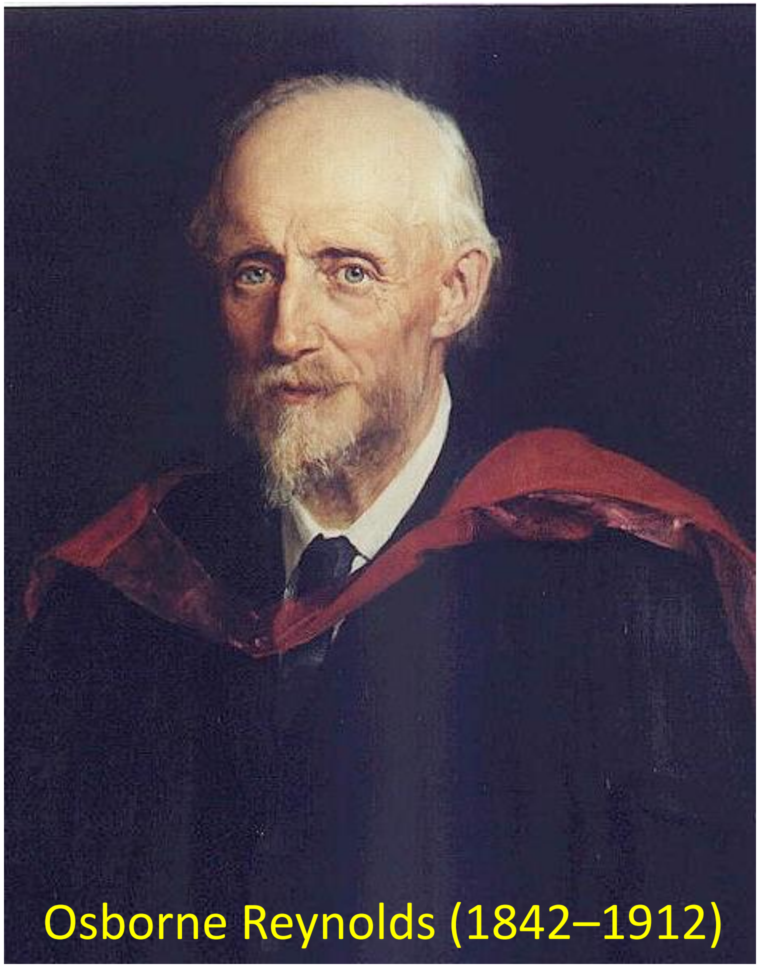
1. $\langle u + v \rangle = \langle u \rangle + \langle v \rangle$;
2. $\langle cu \rangle = c\langle u \rangle$;
3. $\langle c \rangle = c$;
4. $\langle \langle u \rangle v \rangle = \langle u \rangle \langle v \rangle$.

Hence, for $u = \langle u \rangle + u'$,

(i). $\langle \langle u \rangle \rangle = \langle u \rangle$; $\quad \langle u' \rangle = 0$;

(ii). $\langle \langle u \rangle \langle v \rangle \rangle = \langle u \rangle \, \langle v \rangle$;

(iii). $\langle \langle u \rangle v' \rangle = 0$.



Osborne Reynolds (1842–1912)

Warning:

Gaussian smoothing, often used in astronomy, does not satisfy (4),

$$\langle \langle u \rangle \, v \rangle \neq \langle u \rangle \, \langle v \rangle,$$

and hence (i) is not true,

$$\langle \langle u \rangle \rangle \neq \langle u \rangle.$$

Turbulent flows are random

$\Rightarrow$ Velocity $\vec{v}$, pressure p, magnetic field $\vec{B}$, density ρ, etc. are random functions of position $\vec{x}$ and time t.

For a random function $A(x)$, define:

- the average $\langle A \rangle$ and fluctuations a: $A = \langle A \rangle + a, \quad \langle a \rangle = 0$,
- the variance $\sigma_A^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2 = \langle a^2 \rangle$(*),
- the standard deviation (or the root-mean-square value) σ_A.

(*) For the Gaussian smoothing this is not true: $\sigma_A^2 \neq \langle a^2 \rangle$. See Germano, JFM, 238, 325, 1992

2.3. Correlation and structure functions

The autocorrelation function of $A(x)$:
a measure of relation between neighbouring fluctuations:

$$\begin{aligned} C(x_1, x_2) &= \langle a(x_1) a(x_2) \rangle \\ &= \langle [A(x_1) - \langle A \rangle][A(x_2) - \langle A \rangle] \rangle \\ &= \langle A(x_1) A(x_2) \rangle - \langle A \rangle^2 \,, \end{aligned}$$

where $\langle A \rangle$ can depend on x.

$$C(x, x) = \sigma_A^2 \,, \qquad C(x_1 - x_2) \to 0 \quad \text{for } |x_1 - x_2| \to \infty \,.$$

The structure function of $A(x)$: $\quad D(x_1, x_2) = \langle [a(x_1) - a(x_2)]^2 \rangle \,.$

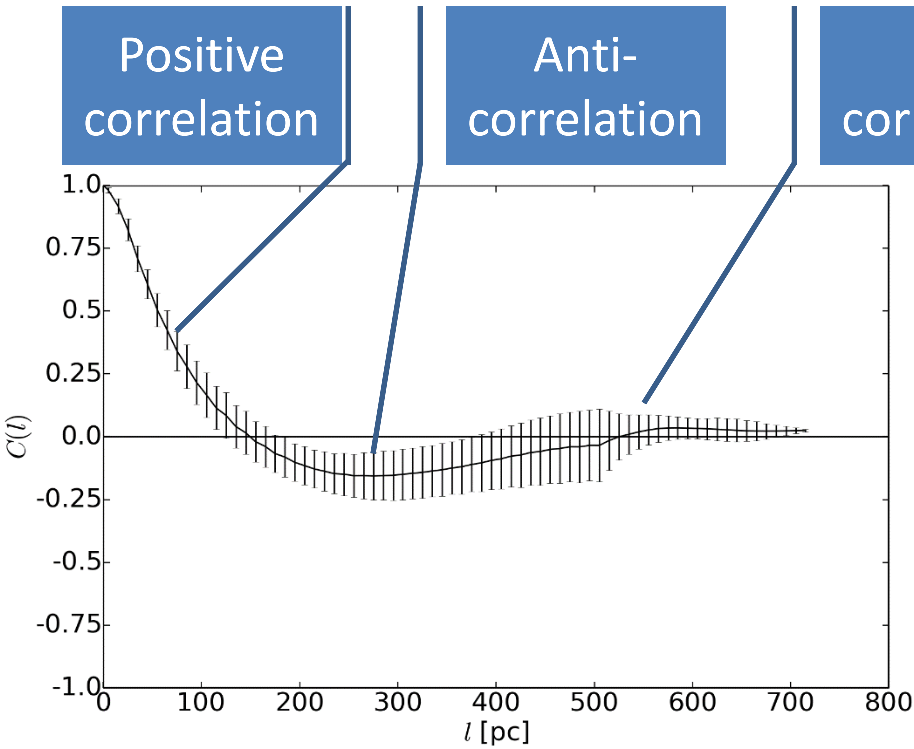
The cross-correlation function of two random functions, $A_1(x)$, $A_2(x)$:

$$B(x_1, x_2) = \langle a_1(x_1) a_2(x_2) \rangle \,.$$

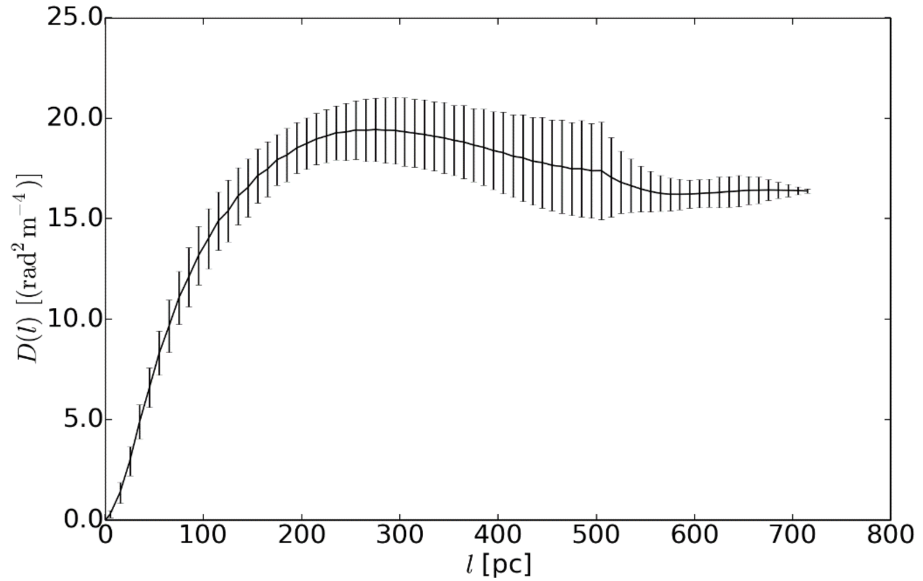
Simulations of interstellar turbulence driven by supernova explosions

(Gent et al., MNRAS, 2013)

$$l = |x_1 - x_2|$$



The autocorrelation function of the Faraday depth



The structure function of the Faraday depth

Structure functions can be calculated from observations or numerical results more accurately and with less computations than autocorrelation functions,

but autocorrelation functions have a more transparent intuitive meaning.

Correlation and structure functions are only useful when applied to random functions.

For a deterministic function $F(x)$,

$$
\begin{aligned}
D(x_1, x_2) &= \langle [F(x) - F(x+l)]^2 \rangle \\
&= [F(x) - F(x+l)]^2 \\
&\approx \left[\frac{\mathrm{d}F}{\mathrm{d}x}(x) \right]^2 l^2 \quad \text{for small } l.
\end{aligned}
$$

It often happens that an observed or computed variable is random at small scales but behaves deterministically at large scales. Then extending its correlation or structure function to those large scales is meaningless and can be misleading.

2.4. Stationary random functions

A random function $A(x)$ is called stationary if its mean value, variance and other statistical properties are independent of x.

- Stationary random functions are *ergodic*

  because different realizations have identical statistical properties.

- Statistical properties of a stationary random function can be obtained from its *single realization*.

- For a stationary random function, with $l = |x_1 - x_2|$:

$$C(x_1, x_2) = C(l)\,, \qquad\qquad B(x_1, x_2) = B(l)\,,$$

$$D(x_1, x_2) = D(l) = 2\langle a^2 \rangle - 2\langle a(x_1) a(x_2) \rangle = 2\left[\sigma_A^2 - C(l)\right].$$

- The correlation length: $l_0 = \dfrac{1}{\sigma_A^2} \displaystyle\int_0^\infty C(l)\,\mathrm{d}l\,.$

- The value of $A(x_2)$ is predictable from $A(x_1)$ when $|x_1 - x_2| \ll l_0$ and unpredictable otherwise: $A(x_1)$ and $A(x_2)$ are uncorrelated if $|x_1 - x_2| \gg l_0$.

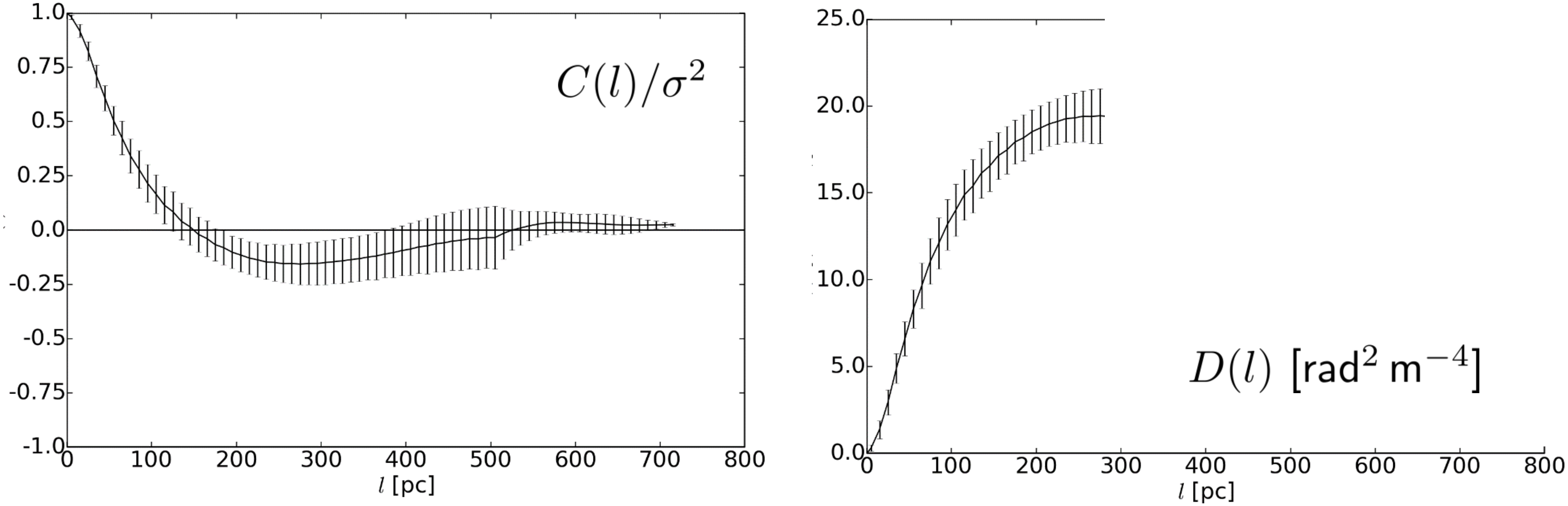
(The most probable weather forecast: tomorrow's weather will be as today's.)

$$D(x_1, x_2) = \langle [a(x_1) - a(x_2)]^2 \rangle \, .$$

Exercise

Show that, for a stationary random function,

$$D(x_1, x_2) \to 2\sigma_A^2 \quad \text{for } |x_1 - x_2| \to \infty \, .$$



It is important to calculate the autocorrelation or structure function for sufficiently large values of the lag l.

2.5. Spectral representation

Power spectrum, or spectral density of $A(x)$:

the Fourier transform of the autocorrelation function:

$$P(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} C(x)\,\mathrm{d}x\,, \qquad C(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} P(x)\,\mathrm{d}k\,.$$

In 3D, $P(\vec{k})$ is called the *3D spectrum*,
the energy spectrum $E(k)$ is obtained by averaging over all directions in the k-space.

In the isotropic case, $P(\vec{k}) = P(k)$,

$$E(k)\,\mathrm{d}k = \frac{1}{4\pi} \int_0^{\pi} \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi P(k) k^2\,\mathrm{d}k\,,$$

$$E(k) = k^2 P(k)\,.$$

2.6. Correlation versus statistical dependence

The cross-correlation function of $A_1(x)$ and $A_2(x)$: $B(x_1, x_2) = \langle a_1(x_1)a_2(x_2)\rangle$.

Correlations of random functions:

- Correlation, $B > 0$: a_1 large where a_2 is large.
- Anticorrelation, $B < 0$: a_1 large where $-a_2$ is large (or A_2 is small).
- No correlation, $B = 0$: $A_1(x)$ and $A_2(x)$ are uncorrelated, and then $\langle A_1 A_2\rangle = \langle A_1\rangle\langle A_2\rangle$.

Statistically independent random functions:
their joint probability density is equal to the product of their respective probability densities, $p(A_1, A_2) = p_1(A_1)p_2(A_2)$.

Statistical independence: the values of one function do not affect the values of the other, and vice versa.

Statistical independence: $p(A_1, A_2) = p_1(A_1)p_2(A_2)$.

- Statistically independent functions are uncorrelated:

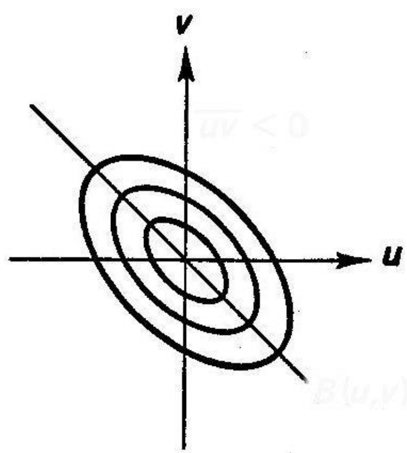
$$\begin{aligned} B(x_1, x_2) &= \langle a_1(x_1) a_2(x_2) \rangle \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1 a_2 p(a_1, a_2)\, \mathrm{d}a_1\, \mathrm{d}a_2 \\ &= \int_{-\infty}^{\infty} a_1 p_1(a_1)\, \mathrm{d}a_1 \int_{-\infty}^{\infty} a_2 p_2(a_2)\, \mathrm{d}a_2 \\ &= 0 \qquad \text{since } \langle a_1 \rangle = \langle a_2 \rangle = 0\,. \end{aligned}$$

- The reverse is not always true: uncorrelated functions are not necessarily statistically independent.

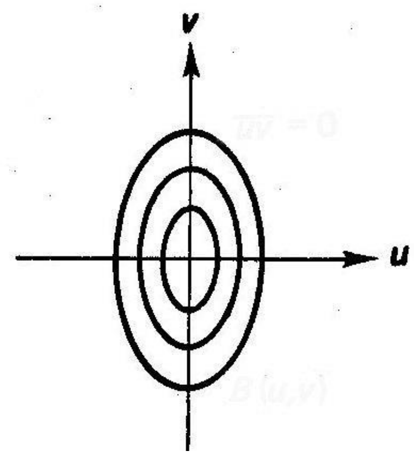
Uncorrelated functions are not necessarily statistically independent.

Contours of joint probability density $p(u,v)$ for random variables u, v that are:

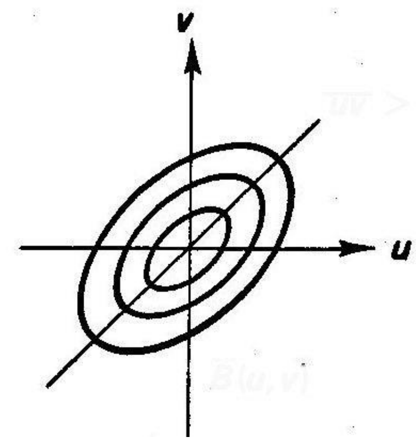
anticorrelated



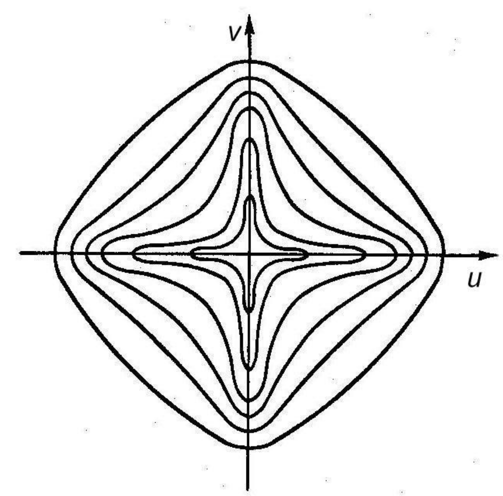
uncorrelated



correlated



Contours of $p(u,v)$ for uncorrelated u, v that tend to inhibit each other, and so are statistically dependent on each other: u and v are seldom large simultaneously.



(some) Things to remember

- **PDF** = probability density function: $p(a)\,\mathrm{d}a =$ the probability that the random variable A takes a value in the interval $(a, a+\mathrm{d}a)$. Effectively, the histogram of a, normalized to the unit area, $\int_{-\infty}^{\infty} p(a)\,\mathrm{d}a = 1$.
- Gaussian smoothing (the convolution of a signal with a Gaussian kernel) does **not** satisfy the Reynolds rules of averaging.
- **Statistically independent** random functions are always uncorrelated (and statistically dependent ones are correlated), but uncorrelated functions can be statistically mutually dependent.
- A **Gaussian random function** [$a(x)$ is a Gaussian random variable at any fixed x] is fully characterized by its (1) mean value, (2) standard deviation and (3) autocorrelation function.
- For a **Gaussian random function**, lack of correlation does imply statistical independence.

3

Phenomenology of fluid turbulence

The Navier–Stokes equation,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}\,,$$

known since 1823, probably contains all of turbulence (and much more), but the nature of turbulence remains one of the most important unsolved problems in physics.

Notation

$\vec{v} = \vec{V} + \vec{u} =$ velocity, $\langle \vec{v} \rangle = \vec{V} =$ mean velocity,

$\rho =$ density, $p =$ pressure, $\nu =$ kinematic viscosity,

$\vec{B} = \vec{B}_0 + \vec{b} =$ magnetic field, $\langle \vec{B} \rangle = \vec{B}_0 =$ mean magnetic field,

$\vec{V}_{\mathrm{A}} = \dfrac{\vec{B}}{\sqrt{4\pi\rho}} =$ Alfvén velocity.

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}\,,$$

Consider a 1D velocity field, $\vec{v} = (v(x), 0, 0)$, neglect pressure, $p = 0$:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}\,,$$ the Burgers equation.

3.1. Energy conservation

$$\mathcal{E} = \tfrac{1}{2}\int_{-\infty}^{\infty} |\vec{v}|^2 \,\mathrm{d}x = \text{kinetic energy per unit mass.}$$

$$\mathcal{E} = \tfrac{1}{2} \int_{-\infty}^{\infty} |\vec{v}|^2 \, \mathrm{d}x$$

Multiply $\dfrac{\partial v}{\partial t} + v \dfrac{\partial v}{\partial x} = \nu \dfrac{\partial^2 v}{\partial x^2}$ by v and integrate over x:

$$v \frac{\partial v}{\partial t} + v^2 \frac{\partial v}{\partial x} = \nu v \frac{\partial^2 v}{\partial x^2} \, ,$$

$$\tfrac{1}{2} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} v^2 \, \mathrm{d}x + \tfrac{1}{3} \int_{-\infty}^{\infty} \frac{\partial v^3}{\partial x} \, \mathrm{d}x = \nu \int_{-\infty}^{\infty} v \frac{\partial^2 v}{\partial x^2} \, \mathrm{d}x \, .$$

Flow confined to a finite domain: $v \to 0$ and $\partial v / \partial x \to 0$ for $x \to \pm\infty$:

$$\int_{-\infty}^{\infty} \frac{\partial v^3}{\partial x} \, \mathrm{d}x \quad = \quad \tfrac{1}{3} v^3 \Big|_{-\infty}^{\infty} = 0 \, ,$$

$$\int_{-\infty}^{\infty} v \frac{\partial^2 v}{\partial x^2} \, \mathrm{d}x \quad = \quad v \frac{\partial v}{\partial x} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\frac{\partial v}{\partial x} \right)^2 \mathrm{d}x = - \int_{-\infty}^{\infty} \left(\frac{\partial v}{\partial x} \right)^2 \mathrm{d}x \, .$$

$$\tfrac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} v^2\,\mathrm{d}x + \tfrac{1}{3}\int_{-\infty}^{\infty} \frac{\partial v^3}{\partial x}\,\mathrm{d}x = \nu \int_{-\infty}^{\infty} v\frac{\partial^2 v}{\partial x^2}\,\mathrm{d}x\ .$$

$$\int_{-\infty}^{\infty} \frac{\partial v^3}{\partial x}\,\mathrm{d}x = 0\ , \qquad \int_{-\infty}^{\infty} v\frac{\partial^2 v}{\partial x^2}\,\mathrm{d}x = -\int_{-\infty}^{\infty}\left(\frac{\partial v}{\partial x}\right)^2\,\mathrm{d}x\ ,$$

$$\mathcal{E} = \tfrac{1}{2}\int_{-\infty}^{\infty} v^2\,\mathrm{d}x\ .$$

The energy equation:

$$\frac{\partial \mathcal{E}}{\partial t} = -\nu S\ , \qquad S = \int_{-\infty}^{\infty}\left(\frac{\partial v}{\partial x}\right)^2\,\mathrm{d}x > 0\ ,$$

kinetic energy is conserved, $\mathcal{E} = \mathrm{const}$, if viscosity vanishes, $\nu = 0$.

νS is the dissipation rate of kinetic energy, related to the shear rate $\dfrac{\partial v}{\partial x}$.

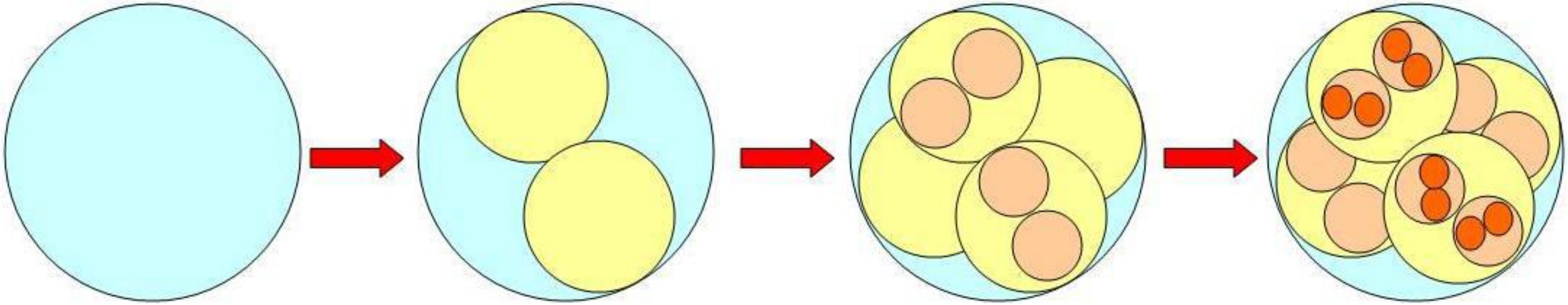
Dissipation rate νS can remain finite even if $\nu \to 0$ because $\dfrac{\partial v}{\partial x} \propto \nu^{-1/2}$.

3.2. Spectral energy transfer

A flow represented by a single Fourier mode initially:

$$v = \sin kx \,, \qquad (\vec{v}\cdot\nabla)\vec{v} = v\frac{\partial}{\partial x}v = k\sin(kx)\cos(kx) \propto \sin(2kx) \,,$$

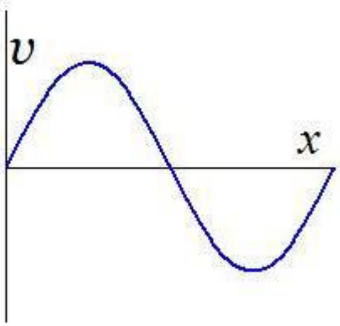
so the inertia force drives small-scale motions, i.e., transfers kinetic energy to small scales, from wavenumber k to $2k$, then from $2k$ to $4k$, etc., resulting in the *energy cascade* in the k-space towards small scales.



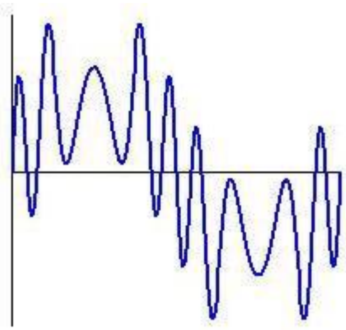
Flow complexity increases with the range of scales involved:

$$v = \sum_{n=0}^{N-1} k_n^{-1/3} \sin(2\pi\, k_n x) \ , \qquad k_n = 2^n$$

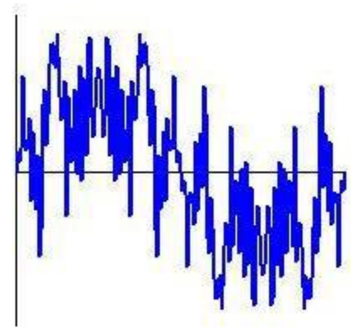
$N = 1$



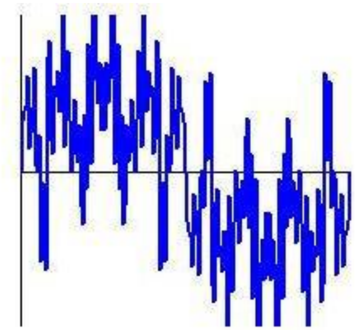
$N = 2$



$N = 3$



$N = 4$



The flow becomes random (for all practical purposes, at least) as soon as the energy cascade produces a sufficiently wide range of scales.

“. . . the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion.”

Leonardo da Vinci (1531)

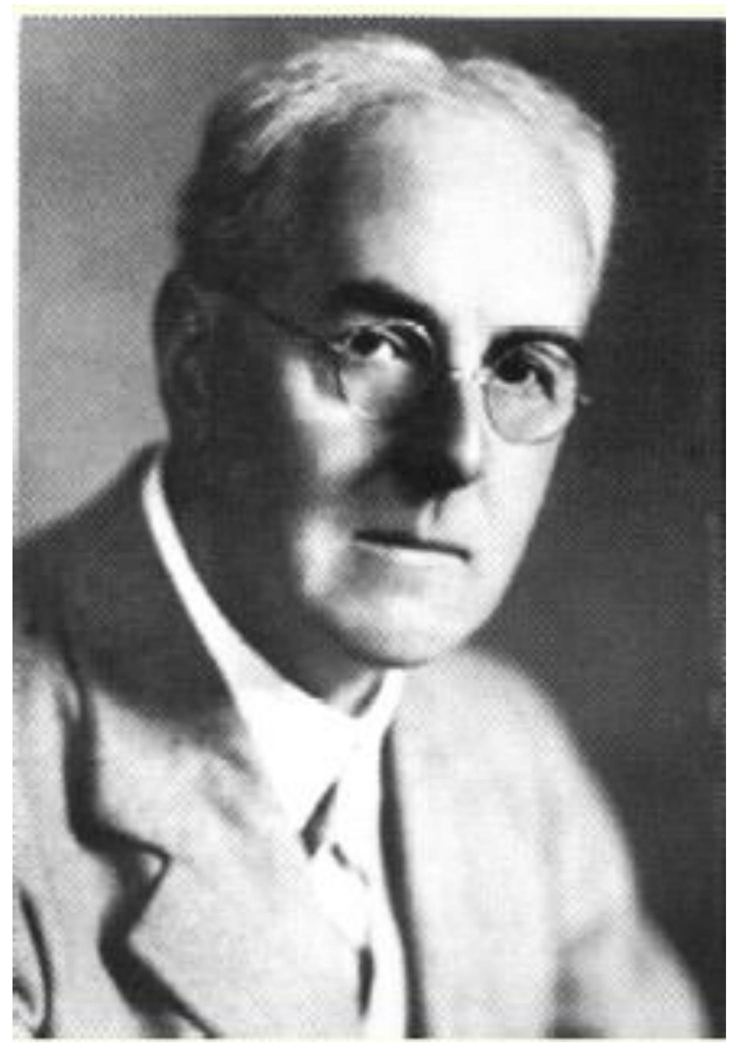
Jonathan Swift (1667–1745):

So, nat'ralists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller yet to bite 'em,
And so proceed *ad infinitum.*
Thus every poet, in his kind,
Is bit by him that comes behind.

Lewis Fry Richardson (1922):

Big whirls have little whirls, which feed on their velocity.

Little whirls have lesser whirls, and so on to viscosity.



1881–1953,
born in Newcastle upon Tyne

A necessary condition for turbulence

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} \ ,$$

The cascade extends over a broad range of k when

$$|(\vec{v} \cdot \nabla)\vec{v}| \gg |\nu \nabla^2 \vec{v}|,$$

that is, when $kv^2 \gg \nu k^2 v$, or

$$\mathrm{Re} = \frac{lv}{\nu} \gg 1 \ ,$$

$l = 2\pi/k$ is the wavelength (or scale) of the motion.

The Reynolds number Re must be large
for a large number of scales to be involved in the motion,
i.e., for a flow to be turbulent.

Free shear layers become turbulent when $\mathrm{Re} > (3-5) \times 10^3$.

In the cool ISM, $\mathrm{Re} = 10^5 - 10^7$ (Elmegreen & Scalo, 2004a),
hence expect the ISM to be turbulent,
if only there are suitable forces to drive the turbulence.

3.1. Kolmogorov's spectrum

Andrey Nikolaevich Kolmogorov (1903–1987)

Consider an incompressible, homogeneous, isotropic turbulent flow, and those scales where viscosity is still unimportant, so energy is conserved.

Spectral description of the turbulent energy cascade:

- $\mathcal{E} = \frac{1}{2}v_0^2 = \int_0^\infty E(k)\,\mathrm{d}k$, specific kinetic energy, $[\mathcal{E}] = \mathrm{cm}^2/\mathrm{s}^2$.

- $E(k)\,dk = \frac{1}{2}v^2(k)\frac{\mathrm{d}k}{k} = \frac{1}{2}v^2(k)\,\mathrm{d}(\ln k)$,

  spectral energy density (or kinetic energy spectrum, or specific kinetic energy per unit interval of $\ln k$).

- $v_0 = \sqrt{2\mathcal{E}}$, the r.m.s. velocity.

- $v(k) = \sqrt{2kE(k)}$, velocity at a wavenumber k,

  $[E(k)] = \mathrm{cm}^3/\mathrm{s}^2$.

Kinetic energy is conserved,
hence all the energy arriving to k is transferred to a larger k:
energy transfer rate along the spectrum is independent of k,

$$\frac{v^2(k)}{\tau} = \varepsilon,$$

$\varepsilon =$ const, energy transfer rate, $\tau =$ time scale of the energy transfer.

$$\tau \simeq \frac{l}{v} = \text{eddy turnover time, at a scale } l, \quad \tau \simeq \frac{1}{kv(k)} \ .$$

$$\frac{v^2(k)}{\tau(k)} = \frac{v^2(k)}{1/[kv(k)]} = kv^3(k) = \varepsilon \ ,$$

resulting in Kolmogorov's spectrum

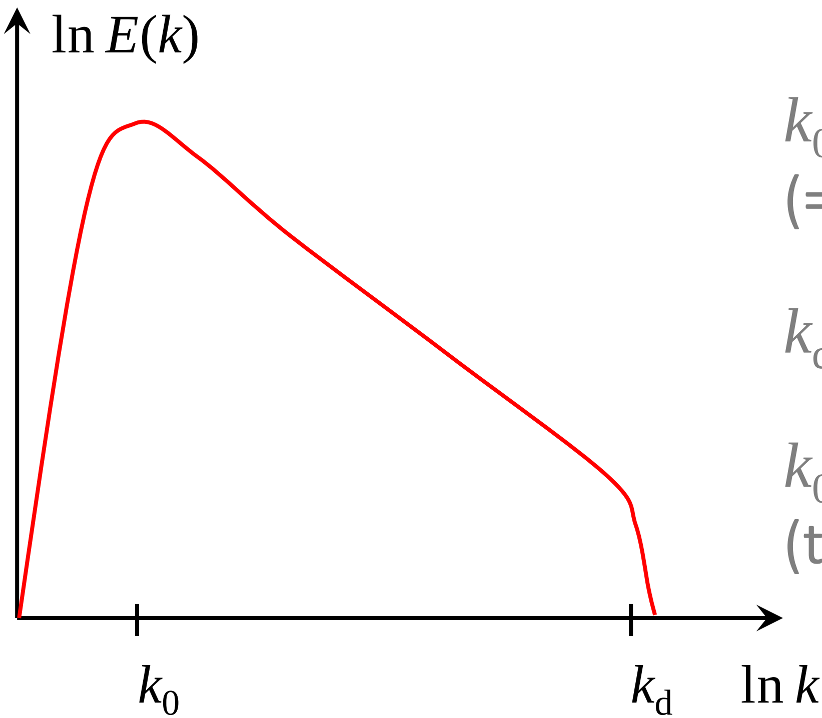
$$\boxed{v(k) = \varepsilon^{1/3} k^{-1/3} \propto l^{1/3} \ , \quad E(k) = k^{-1} v^2(k) = \varepsilon^{2/3} k^{-5/3},}$$

up to a dimensionless constant of order unity.

$$v(k) = \varepsilon^{1/3} k^{-1/3} \propto l^{1/3} \,, \qquad E(k) = k^{-1} v^2(k) = \varepsilon^{2/3} k^{-5/3} \,,$$

$$\tau(k) = \frac{1}{k v(k)} \,.$$

Energy transfer rate: $\varepsilon = \dfrac{v^2(k)}{\tau(k)} = \dfrac{v^2(k_0)}{\tau(k_0)} = k_0 \, v_0^3$



k_0 = energy injection scale
(= integral scale ≈ correlation length),

k_d = dissipation scale,

$k_0 < k < k_d$, the inertial range
(the flow is controlled by inertia forces).

Kinetic energy is injected at $k = k_0$ and cascades to larger k,
to dissipate (be converted into heat) at $k = k_d$.

The turbulent cascade terminates at $k = k_{\rm d}$ such that

$|(\vec{v} \cdot \nabla)\vec{v}| \simeq |\nu \nabla^2 \vec{v}|,$ or $k_{\rm d} v(k_{\rm d})^2 \simeq \nu k_{\rm d}^2 v(k_{\rm d}),$

or $\dfrac{v(k_{\rm d})}{\nu k_{\rm d}} = \mathrm{Re}|_{k=k_{\rm d}} \simeq 1.$

Then $\dfrac{v_0\,(k_{\rm d}/k_0)^{-1/3}}{\nu k_{\rm d}} = 1,$ $\boxed{k_{\rm d} = k_0 \mathrm{Re}^{3/4}}$.

The inertial range becomes broader with Re.

However, small is ν, motions eventually decay in a time

$$\tau_0 \simeq \frac{l_0}{v_0}\ ,$$

the turnover time of the largest eddy.

This is why turbulence requires continuous supply of energy.

The structure function for Kolmogorov's spectrum:

$$v(k) = \varepsilon^{1/3} k^{-1/3} \qquad \text{or} \qquad v(l) = (\varepsilon l)^{1/3}.$$

Then

$$D(l) = \langle [\vec{v}(\vec{x}) - \vec{v}(\vec{x} + \vec{l})]^2 \rangle = \langle v^2(l) \rangle = (\varepsilon l)^{2/3}, \qquad l_\mathrm{d} \ll l \ll l_0,$$

$$l_0 = 2\pi/k_0, \qquad l_\mathrm{d} = 2\pi/k_\mathrm{d}.$$

An important implication:

However small is viscosity, turbulent energy is converted into heat in a short tome of order of eddy turnover time.

Reducing viscosity does not change this but only makes the turbulent spectrum wider

(k_d becomes larger, $l_\mathrm{d} = 2\pi/k_\mathrm{d}$ becomes smaller).

4

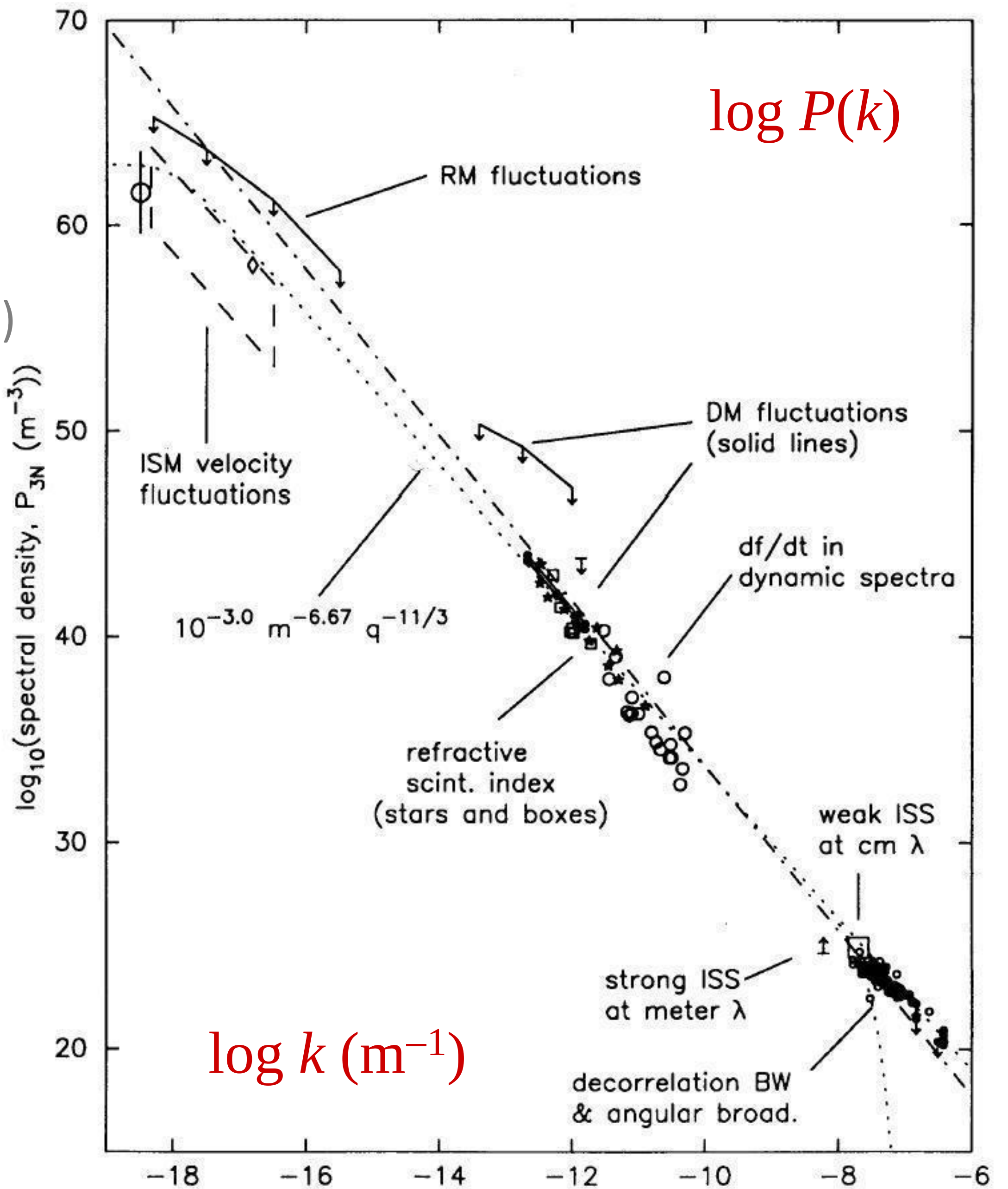
Interstellar turbulence

The Big Power Law in the sky:
3D electron density power spectrum in the ISM
(Armstrong et al., *ApJ*, **433**, 209, 1995)

Kolmogorov's spectrum at $10^{10} < l < 10^{20}\,\mathrm{cm}$.

Turbulence in the ISM:

$$l_0 \simeq 100\,\mathrm{pc},$$
$$v_0 \simeq c_\mathrm{sound} \simeq 10\,\mathrm{km\,s^{-1}},$$
$$\tau_0 \simeq l_0/v_0 \simeq 10^7\,\mathrm{yr}.$$



c_sound = speed of sound in the ISM ($T = 10^4$ K).

4.1. Energy content and energy sources

- Turbulent kinetic energy density: (Mac Low & Klessen 2004)

$$E = \tfrac{1}{2}\rho v_0^2 \simeq 10^{-12}\,\frac{\mathrm{erg}}{\mathrm{cm}^3}\left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)\left(\frac{v_0}{10\,\mathrm{km\,s}^{-1}}\right)^2 .$$

- Magnetic energy density: $M = \dfrac{b_0^2}{8\pi} \simeq E.$

- Energy dissipation rate per unit mass: $\varepsilon \simeq \dfrac{v_0^3}{l_0} \simeq 3\times 10^{-3}\,\mathrm{erg\,g^{-1}\,s^{-1}}$,

  and per unit volume: $\varepsilon_V \simeq \rho\dfrac{v_0^3}{l_0} \simeq 5\times 10^{-27}\,\mathrm{erg\,cm^{-3}\,s^{-1}}$,

- Energy dissipation time $=$ largest eddy turnover time:

$$\frac{E}{\varepsilon_V} = \tau_0 = \frac{l_0}{v_0} = 10^7\,\mathrm{yr}\left(\frac{l_0}{100\,\mathrm{pc}}\right)\left(\frac{v_0}{10\,\mathrm{km\,s}^{-1}}\right)^{-1} .$$

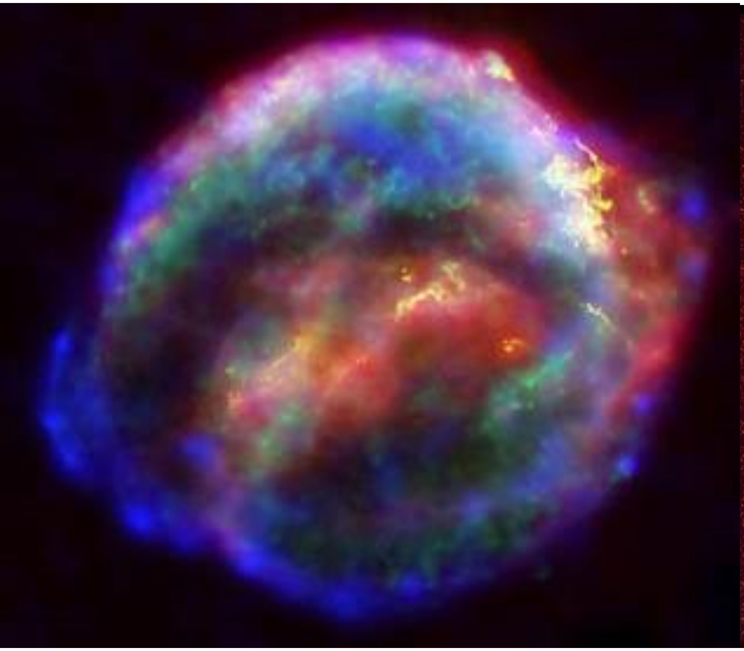
Energy sources of the interstellar turbulence

Driving mechanism	ε_V, $\mathrm{erg\,cm^{-3}\,s^{-1}}$
Supernova explosions	3×10^{-26}
Stellar winds	3×10^{-27}
Protostellar outflows	2×10^{-28}
Stellar ionizing radiation	5×10^{-29}
Galactic spiral shocks	4×10^{-29}
Magneto-rotational instability	3×10^{-29}
H II regions	3×10^{-30}

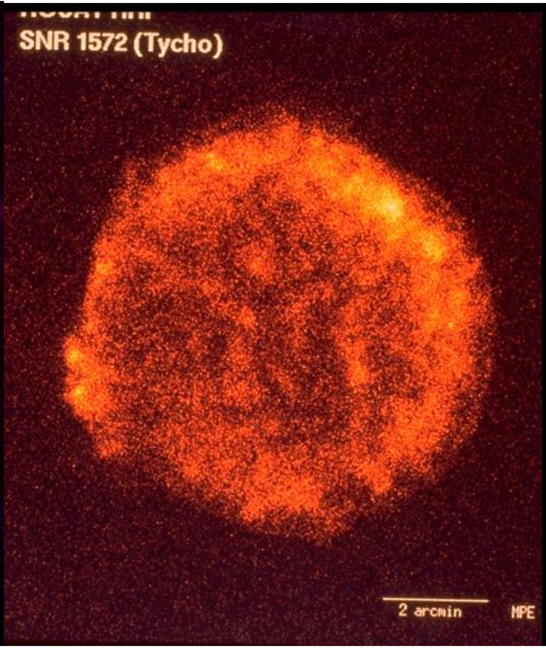
4.2. Turbulence driven by supernovae

Supernova remnants: expanding bubbles
of hot gas, magnetic fields and relativistic particles

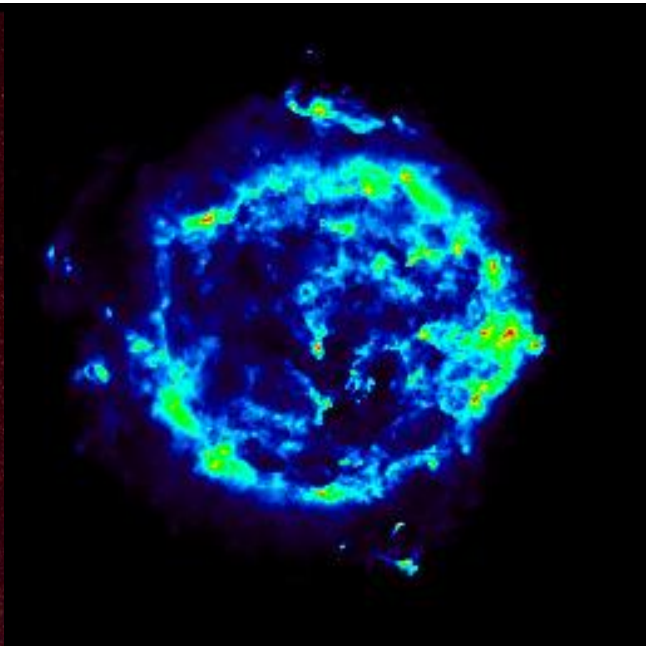
Kepler's SN 1604 (composite)



Tycho SN 1572 (X-rays)



Cas A (radio, λ6 cm)



Wright et al., Astrophys. J. **518**, 284, 1999

SN explosions:

- energy release $E_{\mathrm{SN}} = 10^{51}$ erg per SN event,
- one type II SN per 50 yr near the Sun (frequency $\nu_{\mathrm{SN}} = 0.02\ \mathrm{yr}^{-1}$),
- occur at (quasi) random times and positions.

Supernova blast wave expands at $10^4\ \mathrm{km\ s}^{-1}$ (Mach 10^3 for first 300 yr),
then pressure equilibrium after 10^6 yr,
then a hot gas bubble of 100 pc in size.

Supernova remnants: expanding bubbles of hot gas
that drive motions in the ambient gas
when their expansion speed reduces to the speed of sound
(i.e., when their internal pressure becomes equal to the external pressure).

Total SN energy supply rate:

$$\varepsilon_{\rm SN} = \frac{E_{\rm SN}\,\nu_{\rm SN}}{\mathcal{V}} \simeq 2\times10^{-25}\,{\rm erg\,cm^{-3}\,s^{-1}}\ ,$$

$\mathcal{V} = 2\pi R_* h_* =$ volume of the star-forming Galactic disc,
$R_* = 16\,{\rm kpc}\ ,\ \ h_* = 100\,{\rm pc}.$

The required energy supply:

$$\varepsilon_V \simeq \rho\frac{v_0^3}{l_0} \simeq 5\times10^{-27}\,{\rm erg\,cm^{-3}\,s^{-1}}\ ,$$

$\simeq 3\%$ of the energy supplied by the SNe is sufficient to drive the interstellar turbulence.

Turbulent scale $=$
SNR radius at pressure balance with the ambient medium

Pressure balance: the momentum-conserving (snowplough) phase.

The beginning of the snowplough phase:
(Dyson & Williams, *The Physics of the Interstellar Medium*, IOP, 1997, §7.3.4)

- age, $t_0 = 3.9 \times 10^4\,\mathrm{yr}$,
- SNR radius, $r_0 = 24\,\mathrm{pc}$,
- expansion velocity, $\dot{r}_0 = 250\,\mathrm{km\,s^{-1}}$,
- dense, cool shell of interstellar gas swept up by the SNR.

SNR expansion law in the snowplough phase:

$$r = r_0\left[1 + 4\,\frac{\dot{r}_0}{r_0}(t - t_0)\right]^{1/4}, \qquad \dot{r} = \dot{r}_0\left[1 + 4\,\frac{\dot{r}_0}{r_0}(t - t_0)\right]^{-3/4}.$$

$$r = r_0\left[1 + 4\,\frac{\dot r_0}{r_0}(t - t_0)\right]^{1/4}, \qquad \dot r = \dot r_0\left[1 + 4\,\frac{\dot r_0}{r_0}(t - t_0)\right]^{-3/4}.$$

$r = l_0$ when $\dot r = c_{\rm sound} = 10\,\mathrm{km\,s^{-1}}$:

$$\dot r = c_{\rm sound} \quad \Rightarrow \quad 1 + 4\,\frac{\dot r_0}{r_0}(t - t_0) = \left(\frac{c_{\rm sound}}{\dot r_0}\right)^{-4/3}$$

$$\Rightarrow \quad r = r_0\left(\frac{c_{\rm sound}}{\dot r_0}\right)^{-1/3} \simeq 70\,\mathrm{pc}\ .$$

Conclusion:

the integral scale of the interstellar turbulence is $l_0 = 50$–$100\,$pc

the SNR age when it disintegrates is $t/t_0 \simeq 44$.

Efficiency of SN energy conversion

(Dyson & Williams, *The Physics of the Interstellar Medium*, IOP, 1997, §7.3.6)

$$r = r_0 \left[1 + 4\,\frac{\dot{r}_0}{r_0}(t - t_0)\right]^{1/4}, \qquad \dot{r} = \dot{r}_0 \left[1 + 4\,\frac{\dot{r}_0}{r_0}(t - t_0)\right]^{-3/4}.$$

Kinetic energy of the SNR shell: $E_{\rm shell} = M_{\rm shell}\,\dot{r}^2 = \dfrac{4\pi}{3}\rho_0 r^3 \dot{r}^2$,
$M_{\rm shell}$ = interstellar gas mass (density ρ_0) swept up by the SNR.

$$t \gg t_0 \quad \Rightarrow \quad r \simeq \left(4 r_0^3 \dot{r}_0 t\right)^{1/4}, \quad \dot{r} \simeq \left(\tfrac{1}{4} r_0^3 \dot{r}_0 t^{-3}\right)^{1/4}$$

Efficiency of energy conversion:

$$\frac{E_{\rm shell}}{E_{\rm SN}} \simeq \frac{\pi}{3\sqrt{2}E_{\rm SN}}\rho_0 r_0^{15/4}\,\dot{r}_0^{5/4} t^{-3/4} \simeq 1.2\left(\frac{t_0}{t}\right)^{3/4} \simeq 8\%, \qquad \frac{t}{t_0} = 44.$$

Conclusions

- SNe are the most important source of interstellar turbulence;
- the correlation scale of the turbulence is $l_0 = 50\text{–}100\ \mathrm{pc}$;
- the turbulent speed is comparable to the speed of sound in the ISM, $v_0 \simeq 10\ \mathrm{km\ s^{-1}}$, or can even exceed it.
- Interstellar turbulence is transonic or supersonic, hence highly compressible, producing strong density fluctuations.

4.3. Observational signatures of interstellar turbulence

- Spectral line broadening by Doppler shifts:

$$\Delta\nu_{\mathrm{D}} = \nu_0 \left(\underbrace{\frac{2k_{\mathrm{B}}T}{m_a c^2}}_{\text{thermal}} + \underbrace{\frac{2v_0^2}{3c^2}}_{\text{turbulent}} \right)^{1/2} ,$$

$\nu_0 =$ central line frequency,
$k_{\mathrm{B}} =$ Boltzmann's constant,
$m_a =$ the emitting atom's mass,
$c =$ speed of light.

Velocity dispersion of interstellar gas scales with the region size l

(Larson, *MNRAS,* **186**, 479, 1979; **194**, 809, 1981)

$$\delta v\ (\mathrm{km\,s^{-1}}) \simeq 1.1 \left(\frac{l}{1\,\mathrm{pc}}\right)^{\beta} , \qquad \beta = 0.4 \pm 0.1$$

consistent with Kolmogorov's law $v(l) \propto l^{1/3}$.

However, the interpretation of the scaling is controversial

(e.g., Mac Low & Klessen 2004)

More recently: various statistical studies of velocity and density fluctuations, especially in molecular clouds

(Elmegreen & Scalo 2004a)

- Radio wave scattering at electron density fluctuations

  $\Rightarrow$ scintillation, pulse broadening of pulsar emission

  Density fluctuations in weakly compressible turbulence:

$$\frac{\delta n}{\langle n \rangle} \simeq \frac{\delta v}{v_{\mathrm{A0}}} \,,$$

  density fluctuations have the same power spectrum as v.

  Significant effects at small scales, $l \lesssim 10^{15}\,\mathrm{cm}$.

4.4. The role of turbulence in the iSM

- Cloudy structure of the interstellar gas produced by compression

Standard picture:

Field, Goldsmith & Habing (1969): cold ($T \simeq 10^2\,\mathrm{K},\ n \simeq 10\,\mathrm{cm}^{-3}$) and warm ($T \simeq 10^4\,\mathrm{K},\ n \simeq 0.1\,\mathrm{cm}^{-3}$) phases in pressure balance.

McKee & Ostriker (1977): cold ($T \simeq 10^2\,\mathrm{K},\ n \simeq 10\,\mathrm{cm}^{-3}$), warm ($T \simeq 10^4\,\mathrm{K},\ n \simeq 0.1\,\mathrm{cm}^{-3}$) and hot ($T \simeq 10^6\,\mathrm{K},\ n \simeq 10^{-3}\,\mathrm{cm}^{-3}$) phases in pressure balance;
hot phase created by uncorrelated SNe: filling factor 70%.

The effect of supersonic turbulence on the ISM: new insights

- Filamentary structure of the cold gas results from shocks.
- H I clouds are formed by compression rather than by thermal & gravitational instabilities $\Rightarrow$ implications for star formation criteria.

- Control of star formation (Mac Low & Klessen 2004)
- Mixing of interstellar gas
  $\Rightarrow$ small (5–20%) metallicity variations in stars and interstellar gas
  (Scalo & Elmegreen 2004b)
- Turbulent pressure

$$v_0 \gtrsim c_{\rm sound} \quad \Rightarrow \quad P_{\rm t} \gtrsim P_{\rm therm} \quad \Rightarrow \quad \text{thicker gas layer.}$$

- Generation of magnetic fields
  - the fluctuation dynamo
    $\Rightarrow$ random, filamentary magnetic fields;
  - the large-scale dynamo
    $\Rightarrow$ magnetic field ordered at the galactic scale
    + volume-filling random magnetic field

5

Alfvén wave turbulence

Interstellar medium is magnetized, with energy density of magnetic field comparable to the kinetic energy density of turbulence,

$$\tfrac{1}{2} v_0^2 \simeq \frac{1}{8\pi} b_0^2 \simeq 10^{-12}\,\mathrm{erg\,cm^{-3}} = 1\,\mathrm{eV\,cm^{-3}}\ .$$

Furthermore,

$$R_{\mathrm{m}} \gg \mathrm{Re} \gg 1\ , \qquad R_{\mathrm{m}} = \frac{lv}{\eta}\ ,$$

$R_{\mathrm{m}} =$ magnetic Reynolds number,

$\eta =$ magnetic diffusivity, $[\eta] = \mathrm{cm^2\,s^{-1}}$.

Thus, interstellar turbulence is a magnetohydrodynamic turbulence

(likewise, for the solar/stellar wind turbulence, turbulence in radio galaxies and quasars, etc.)

5.1. Isotropic Alfvén wave turbulence

An ensemble of Alfvén waves, randomness from their nonlinear interactions, incompressible.

Governing equations: The Navier-Stokes equation and the induction equation for magnetic field:

$$\frac{\partial \vec{\mathbf{B}}}{\partial t} = \underbrace{\nabla \times (\vec{\mathbf{V}} \times \vec{\mathbf{B}})}_{\text{advection, stretching, compression}} + \underbrace{\eta \nabla^2 \vec{\mathbf{B}}}_{\text{diffusion, decay}} .$$

A convenient variable, Alfvén speed at a scale $l = 2\pi/k$:

$$v_{\mathrm{A}}(k) = \frac{b(k)}{\sqrt{4\pi\rho}} .$$

Kinetic energy density: $E = \frac{1}{2}\rho v^2$

Magnetic energy density: $M = \frac{1}{2}\rho v_{\mathrm{A}}^2$

Iroshnikov, *Sov. Astron.*, 7, 566,1964;
Kraichnan, *Phys. Fluids*, 8, 1385, 1965

Kolmogorov turbulence versus isotropic Alfvén wave turbulence

Fluid turbulence	Alfvén wave MHD turbulence
Specific kinetic energy: $\mathcal{E} = \frac{1}{2}v_0^2$	Specific magnetic energy: $\mathcal{M} = \frac{1}{2}v_{\mathrm{A}0}^2$
Kinematic viscosity ν	Magnetic viscosity η
Kinetic energy spectrum: $E(k) = \frac{1}{2}k^{-1}v^2(k)$	Magnetic energy spectrum: $M(k) = \frac{1}{2}k^{-1}v_{\mathrm{A}}^2(k)$
Turbulent eddy	Alfvén wave riding on magnetic field of the largest scale
Constant spectral energy flux: $\dfrac{v^2(k)}{\tau(k)} = \varepsilon$	Constant spectral energy flux: $\dfrac{v^2(k)}{\tau_m(k)} = \dfrac{v_{\mathrm{A}}^2(k)}{\tau_m(k)} = \varepsilon$
Spectral energy transfer rate: $\tau(k) = \dfrac{1}{kv(k)}$	Spectral energy transfer rate: $\tau_m(k) = \tau(k)\dfrac{\tau(k)}{\tau_{\mathrm{A}}(k)}$, $\tau_{\mathrm{A}}(k) = \dfrac{1}{kV_{\mathrm{A}}}$, interaction time of Alfvén waves. $V_{\mathrm{A}} \geq v_0 \Rightarrow \tau_{\mathrm{A}}(k) < \tau(k)$, weak interaction

Kolmogorov turbulence versus isotropic Alfvén wave turbulence

(continued)

Kolmogorov's spectrum: $E(k) = \varepsilon^{2/3} k^{-5/3}$	Iroshnikov–Kraichnan spectrum: $v(k) = v_{\mathrm{A}}(k) = (v_{\mathrm{A}0}\varepsilon)^{1/4} k^{-1/4}$, $E(k) = M(k) = (v_{\mathrm{A}0}\varepsilon)^{1/2} k^{-3/2}$, equipartition between kinetic and magnetic energies, as in a single Alfvén wave.
Dissipation scale: $\tau(k_{\mathrm{d}}) = \nu k_{\mathrm{d}}^2 \quad \Rightarrow \quad k_{\mathrm{d}} = k_0 \mathrm{Re}^{3/4}$	Dissipation scale: $\tau_m(k_{\mathrm{dm}}) = \eta k_{\mathrm{dm}}^2 \Rightarrow k_{\mathrm{dm}} = k_0 R_{\mathrm{m}}^{2/3} \left(\frac{v_0}{v_{\mathrm{A}0}}\right)^{1/3}$. $k_{\mathrm{d}} \ll k_{\mathrm{dm}}$ if $R_{\mathrm{m}} \gg \mathrm{Re}$ (magnetic spectrum extends to smaller scales than velocity spectrum).

5.2. Anisotropic Alfvén wave turbulence

(Sridhar & Goldreich, *ApJ*, **432**, 612, 1994;
Goldreich & Sridhar, *ApJ*, **438**, 763, 1995)

Magnetic field at larger scales introduces anisotropy:

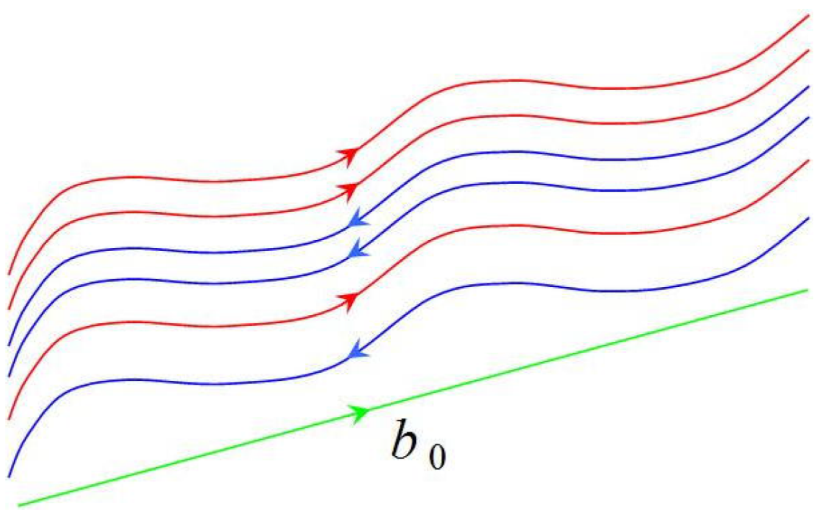
motion along $\vec{\mathrm{b}}_0$ is free, but that across $\vec{\mathrm{b}}_0$ is hindered

$\Rightarrow$ slow variations along the field are allowed,

but the wavelength can be shorter across the field:

$k_\perp \gg k_{||}$, $\perp \, (||) =$ perpendicular (parallel) to $\vec{\mathrm{b}}_0$

$\Rightarrow$ turbulent cells are elongated along $\vec{\mathrm{b}}_0$.



Balance of energy transfer rates across and along $\vec{\mathrm{b}}_0$:

$$\underbrace{k_\perp v_\mathrm{A}(k_\perp)}_{1/\tau(k_\perp)} \simeq \underbrace{k_\| v_{\mathrm{A}0}}_{\|\ \text{wave frequency}}$$

Spectral energy cascade mainly occurs in the $k_\perp$-plane, with

$$\varepsilon \simeq \frac{v_\mathrm{A}^2(k_\perp)}{\tau(k_\perp)} = k_\perp v_\mathrm{A}^3(k_\perp) \ .$$

Combine the two equations to obtain the aspect ratio of the turbulent cells:

$$l_\| \simeq \frac{v_{\mathrm{A}0}}{\varepsilon^{1/3}} l_\perp^{2/3} \simeq l_0^{1/3} l_\perp^{2/3} \ , \qquad \frac{k_\perp}{k_\|} \simeq (l_0 k_\perp)^{1/3} \ ,$$

with

$$l_0 = \frac{v_{\mathrm{A}0}^3}{\varepsilon} \ .$$

$\Rightarrow$ the spectral anisotropy increases with $k_\perp$.

The resulting energy spectrum in the inertial range:

$$E(k_\perp) \ = \ \varepsilon^{2/3} k_\perp^{-5/3} = \left(\frac{v_{\mathrm{A}0}^{\ 3}}{l_0} \right)^{2/3} k_\perp^{-5/3} \ ,$$

$$E(k_\parallel) \ = \ \varepsilon^{3/2} v_{\mathrm{A}0}^{\ -5/2} k_\parallel^{-5/2} \ .$$

6

Dynamos

Dynamo action:

conversion of kinetic energy of a fluid or plasma flow into magnetic energy *without any externally driven electric currents.*

Mirror symmetry, vectors and pseudovectors

Fig. 52–2. A step in space and its mirror image.

Fig. 52–3. A rotating wheel and its mirror image. Note that the angular velocity "vector" is not reversed in direction.

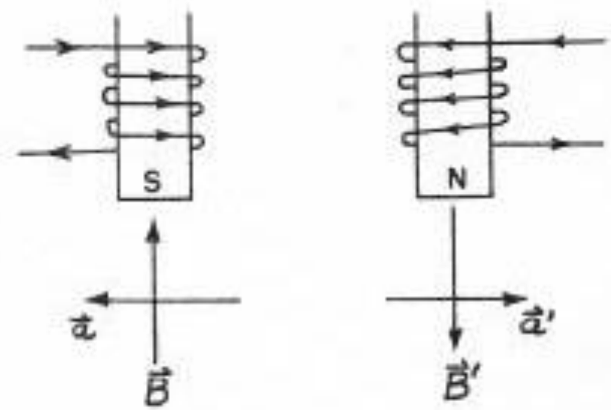


Fig. 52–4. A magnet and its mirror image.

When mirrored, a *polar* vector changes its head, just as the whole space turns inside out.
An axial vector (pseudovector) changes in a very different way: it is usually *reversed in respect to the geometry of the whole space.*

R. Feynman, Lecture Notes in Physics, https://readingfeynman.org/2014/05/09/cpt-symmetry-ii/

- Angular velocity and magnetic field are similar in that both are pseudovectors

- To produce a magnetic field at a scale similar to the size of a physical system, the system must break its mirror symmetry.

- Conclusion: to produce a large-scale magnetic field, any system must be asymmetric with respect to mirror reflection.

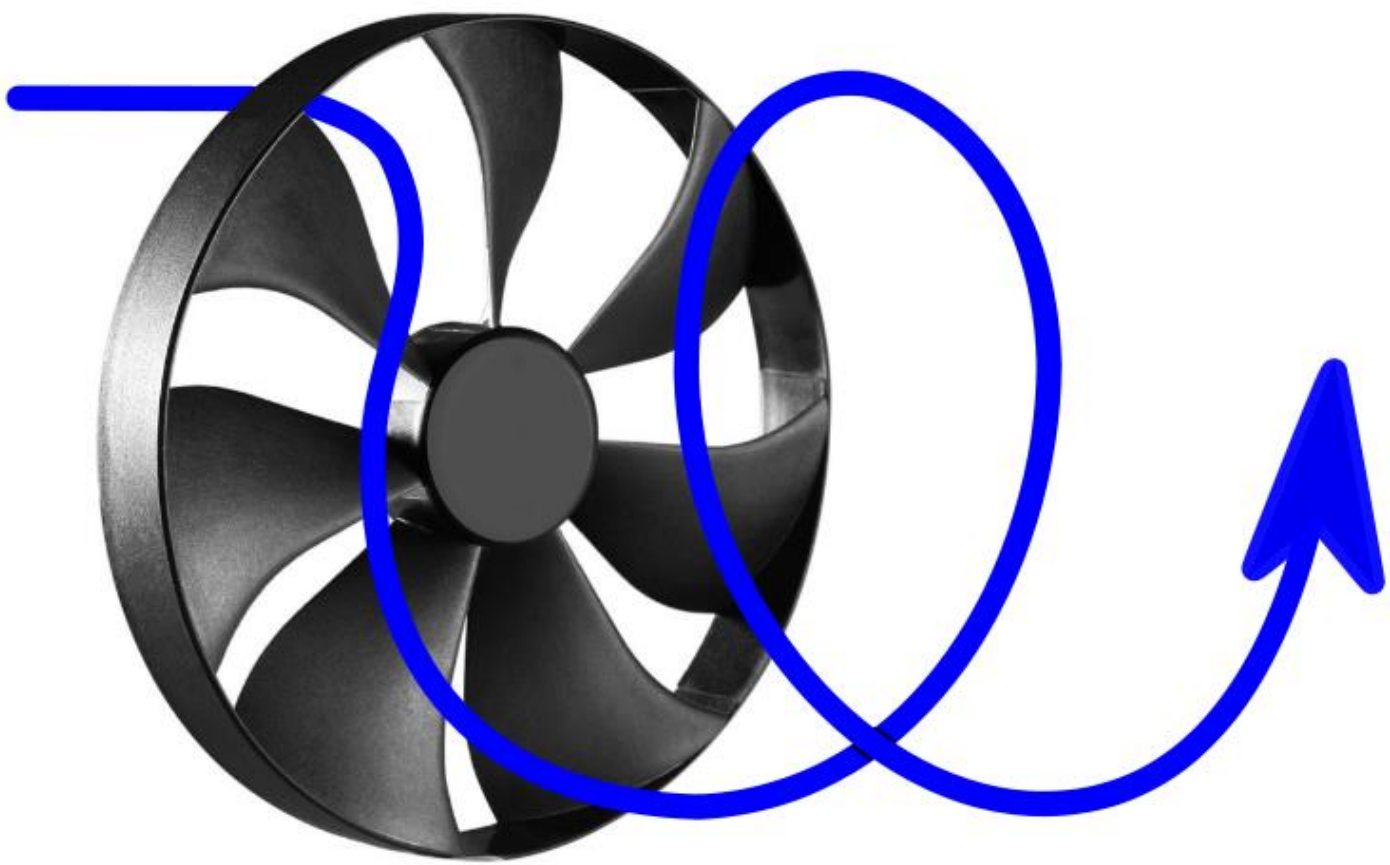
- The most widespread asymmetry is due to rotation.

- If $\vec{a}$ and $\vec{b}$ are polar vectors,
  - $\vec{a} + \vec{b}$ and $\vec{v} = \dot{\vec{a}}$ are polar vectors, $\vec{a} \cdot \vec{b}$ is a scalar,
  - but $\vec{a} \times \vec{b}$ and $\nabla \times \vec{a}$ are axial (pseudo-)vectors and $\vec{a} \times \vec{b} \cdot \vec{c}$ and $\vec{a} \cdot \nabla \times \vec{b}$ are pseudoscalars.
  - Pseudoscalars change sign under mirror reflection.

Helical flow $\vec{v}$ is NOT mirror symmetric

(when reflected, the right-hand screw becomes the left-hand one),

$$\vec{v} \cdot \nabla \times \vec{v} \neq 0.$$



6.1. The induction equation and dynamo action

From Maxwell's equations (for non-relativistic motions) and Ohm's law:

$$\frac{\partial \vec{B}}{\partial t} = \underbrace{\overset{\text{vector}}{\nabla} \times \overset{\text{vector}}{(\underset{\text{vector}}{\vec{v}} \times \underset{\text{pseudo vector}}{\vec{B}})}}_{\text{pseudovector}} + \eta \nabla^2 \vec{B}, \quad \nabla \cdot \vec{B} = 0.$$

$\vec{v}$ = velocity field,

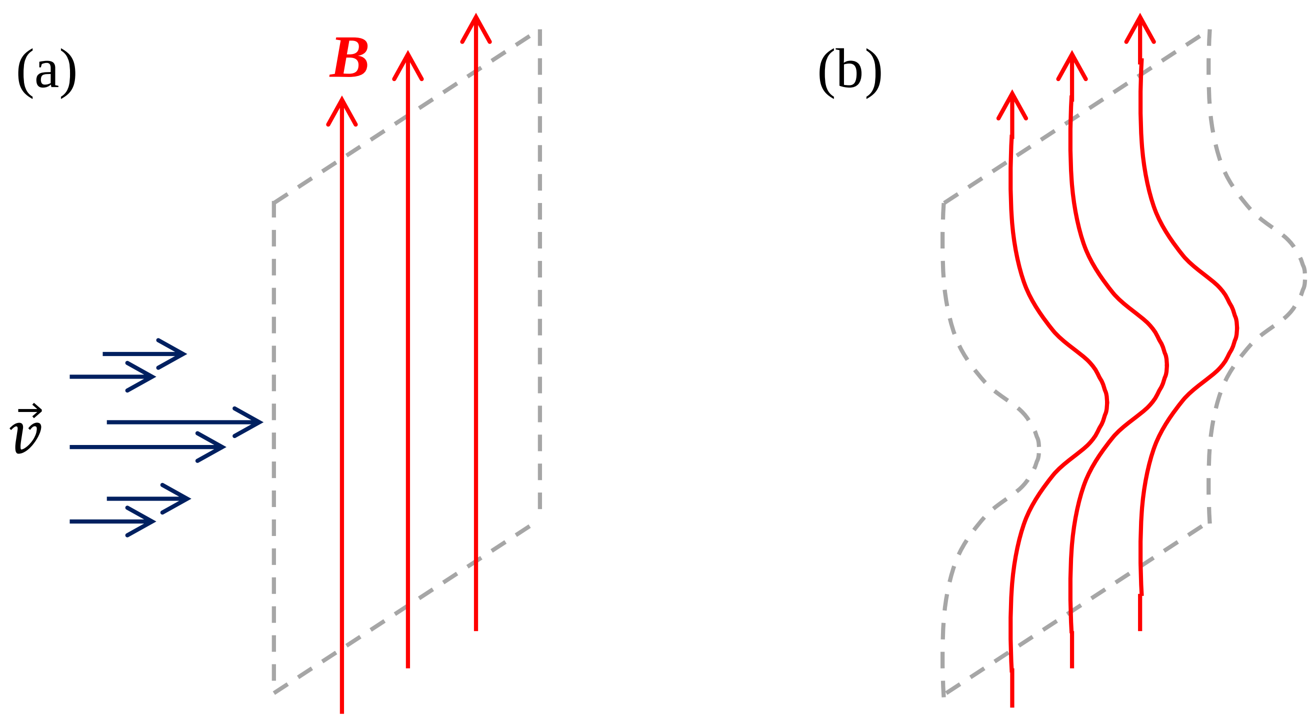
$\vec{B}$ = magnetic field,

η = magnetic diffusivity.

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B}, \quad \nabla \cdot \vec{B} = 0.$$

Equivalently,

$$\frac{\partial \vec{B}}{\partial t} + \underbrace{(\vec{v} \cdot \nabla)\vec{B}}_{\text{advection}} = \underbrace{(\vec{B} \cdot \nabla)\vec{v}}_{\text{stretching}} + \eta \nabla^2 \vec{B}, \qquad \nabla \cdot \vec{B} = 0$$



$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B}.$$

Suppose that $\vec{B}$ is weak ($B^2/8\pi \ll \rho v^2/2$). Then velocity field does not depend on magnetic field and the induction equation has solutions of the form $B = B_\mathrm{i} \mathrm{e}^{\gamma t}$

$B_\mathrm{i} = B|_{t=0}$, the initial (seed) magnetic field;
$\mathrm{Re}\,\gamma$ = the rate of growth ($\mathrm{Re}\,\gamma > 0$) or decay ($\mathrm{Re}\,\gamma < 0$);
$\mathrm{Im}\,\gamma$ = the oscillation frequency.

If $\vec{v} = 0$, $\gamma \simeq -\eta/L^2$: Ohmic decay.

Under what conditions would magnetic field grow, $\mathrm{Re}\,\gamma > 0$?
Equivalently, under what conditions would the dynamo work?

$$|\nabla \times (\vec{v} \times \vec{B})| \gg \eta |\nabla^2 \vec{B}|.$$

$$\frac{vB}{L} \gg \eta \frac{B}{L^2},$$

L is the scale of the magnetic field.

$$R_{\mathrm{m}} = \frac{vL}{\eta} \gg 1,$$

R_{m} is the *magnetic Reynolds number.*

This is a necessary condition for the dynamo action: magnetic field can only decay if $R_{\mathrm{m}} < 1$.

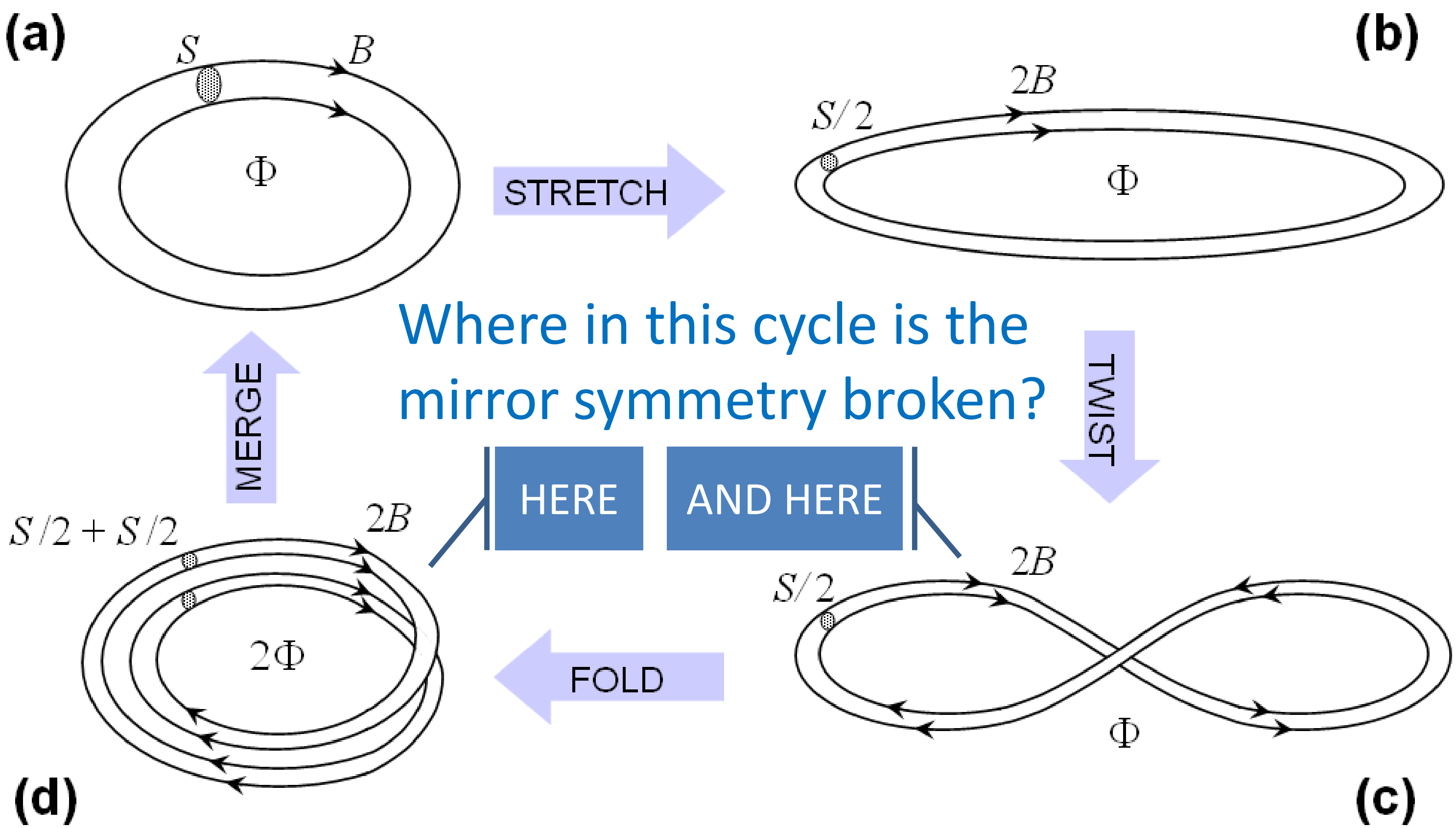
A necessary condition: further conditions need to be met by a successful dynamo.

Magnetic Reynolds numbers of some astrophysical objects

	u_{rms} [cm s^{-1}]	L [cm]	R_{m}
Solar CZ (upper part)	10^6	10^8	10^6
Solar CZ (lower part)	10^4	10^{10}	10^9
Protostellar discs	10^5	10^{12}	10
CV discs and similar	10^5	10^7	10^4
AGN discs	10^5	10^9	10^{11}
Galaxy	10^6	10^{20}	(10^{18})
Galaxy clusters	10^8	10^{23}	(10^{29})

A. Brandenburg, K. Subramanian / Physics Reports 417 (2005) 1–209

6.2. Stretch-Twist-Fold: a conceptual dynamo (Zeldovich's rope dynamo)



B doubles after each cycle t_0: $B \propto 2^n \propto \exp(\gamma t)$, $\gamma = t_0^{-1} \ln 2$.

Yakov Borisovich Zeldovich, IAU Symp., Prague, 1968

To be a dynamo, the flow must be three-dimensional:

2D flows cannot be a dynamo (it's one of antidynamo theorems

– see an excellent discussion of Chris Jones, Section 1.8,

http://www1.maths.leeds.ac.uk/~cajones/LesHouches/chapter.pdf)

6.3. The necessity of dynamos

- Can galactic magnetic fields be primordial?
- Do they need to be maintained by any ongoing dynamo action?

Magnetic fields in a highly conducting turbulent medium

"If $R_\mathrm{m} >> 1$, magnetic field decays only slowly and so does not necessarily need to be continuously maintained" – TRUE?

Wrong, if the system is turbulent:
magnetic energy is transferred along the spectrum and then dissipates in a time of order l_0/v_0, and this time is much shorter than the Ohmic decay time l_0^2/η when $R_\mathrm{m} = l_0 v_0/\eta \gg 1$.

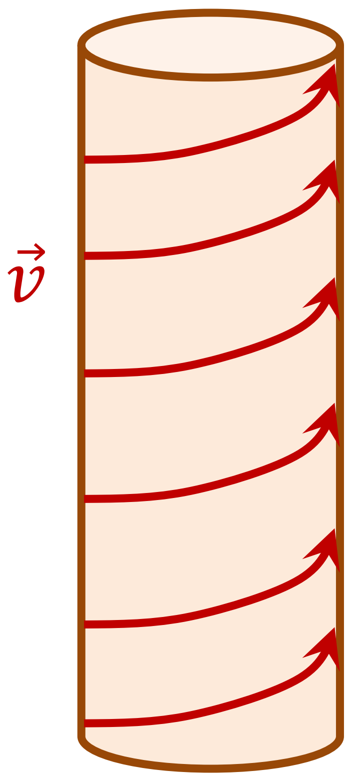
Even without turbulence, a sufficiently strong random magnetic field would drive random motions, and they will dissipate viscously to drain magnetic energy.

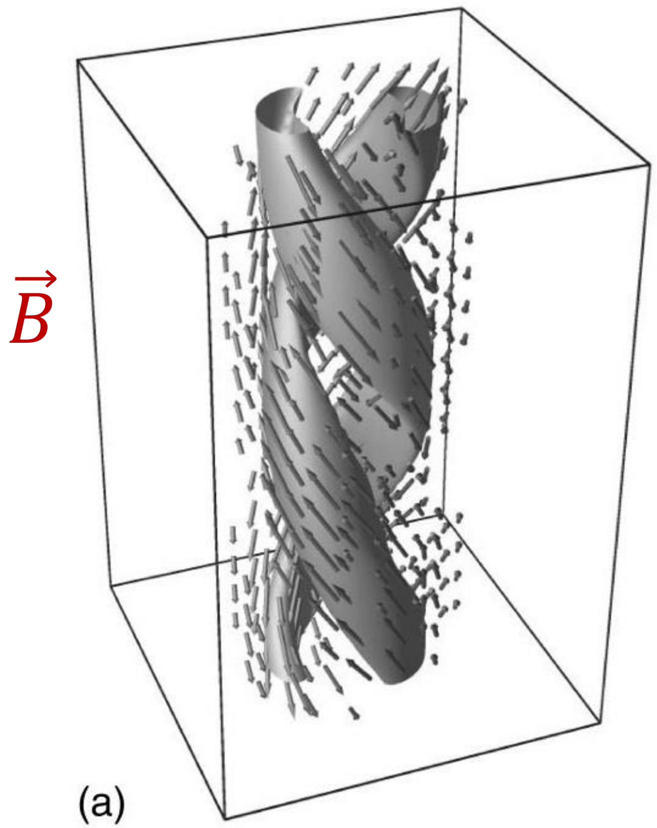
Conclusion: any (3D, MHD) magnetised, turbulent system must host a dynamo (unless the magnetic field is driven by external currents or decays).

6.4. Classification of dynamos

- **Laminar dynamos**: the velocity field is laminar (deterministic).

Example: a swirling flow (as in a AGN jet): the Ponomarenko dynamo



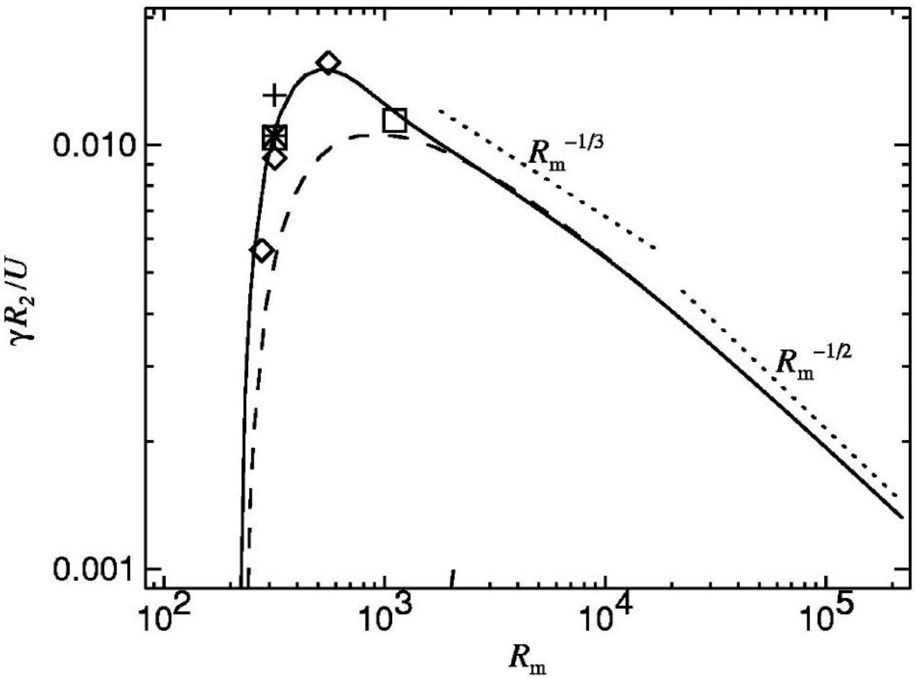


Dobler et al., PRE, 65, 036311, 2002

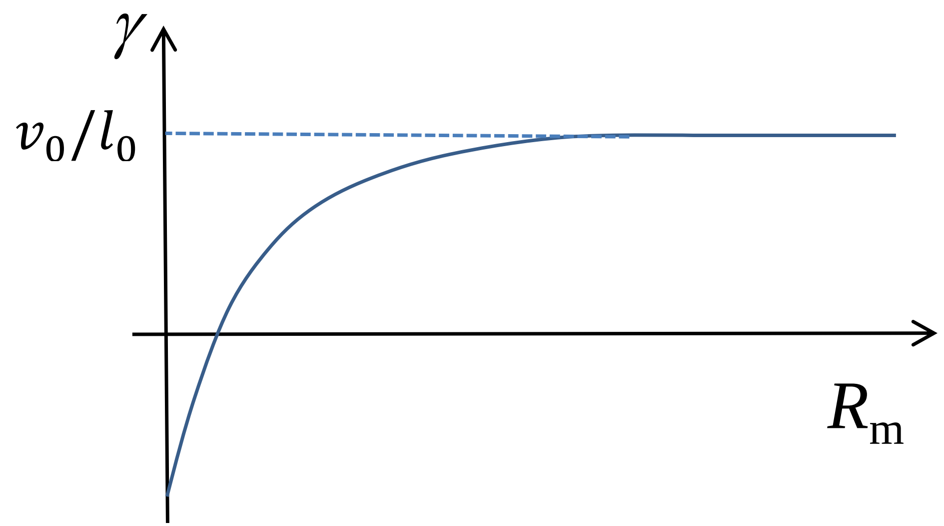
- **Turbulent dynamos**: the velocity field is random (not necessarily turbulent).
    - Fluctuation (small-scale) dynamo: random flow generates random magnetic field (whose scale does not exceed the scale of the flow).
    - Mean-field (large-scale) dynamo: random flow generates a mean magnetic field (whose scale exceeds the scale of the flow).

Astrophysical objects: $R_\mathrm{m} \gg 1$

- **Fast** dynamos: $\mathrm{Re}\,\gamma \to \mathrm{const}$ as $R_\mathrm{m} \to \infty$.
- **Slow** dynamos: $\mathrm{Re}\,\gamma \to 0$ as $R_\mathrm{m} \to \infty$.



The Ponomarenko dynamo



The fluctuation dynamo

All known laminar dynamos are slow: main interest in fast dynamos

6.4. The fluctuation dynamo

A random magnetic field grows if

(1) the flow is random (e.g., turbulent) and

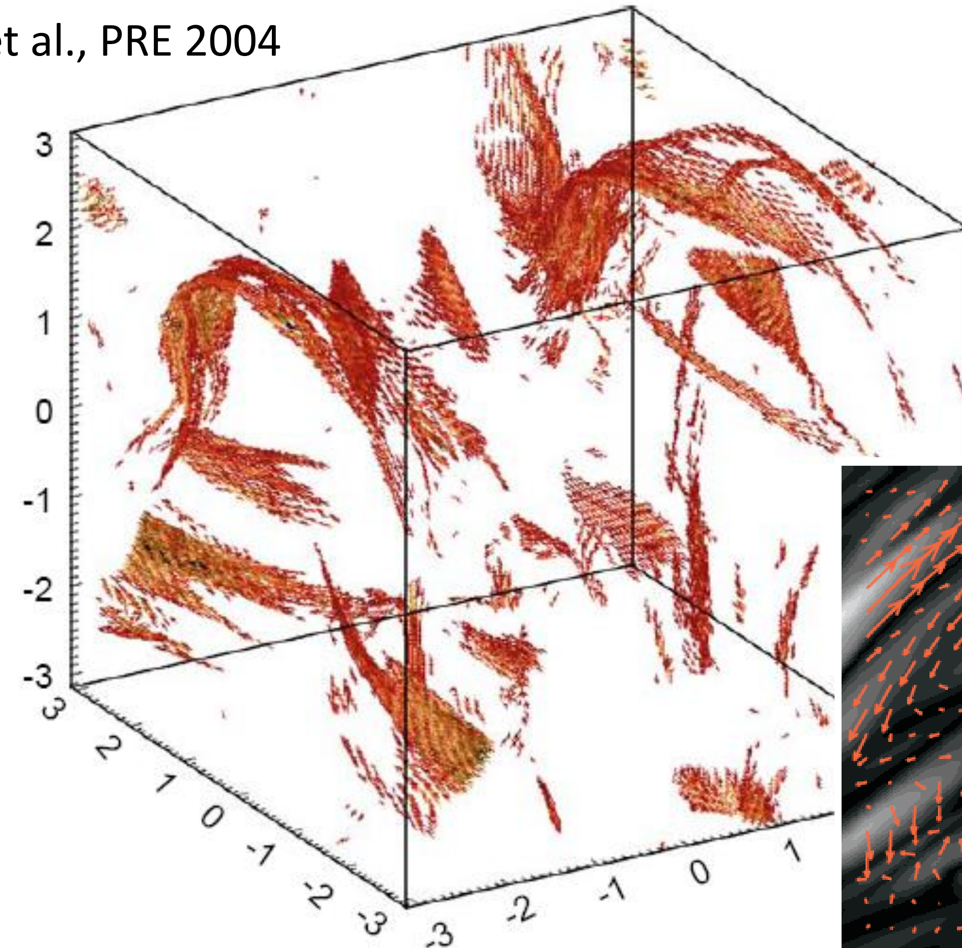
(2) $R_{\mathrm{m}} > R_{\mathrm{m,cr}} \cong 10^2$;

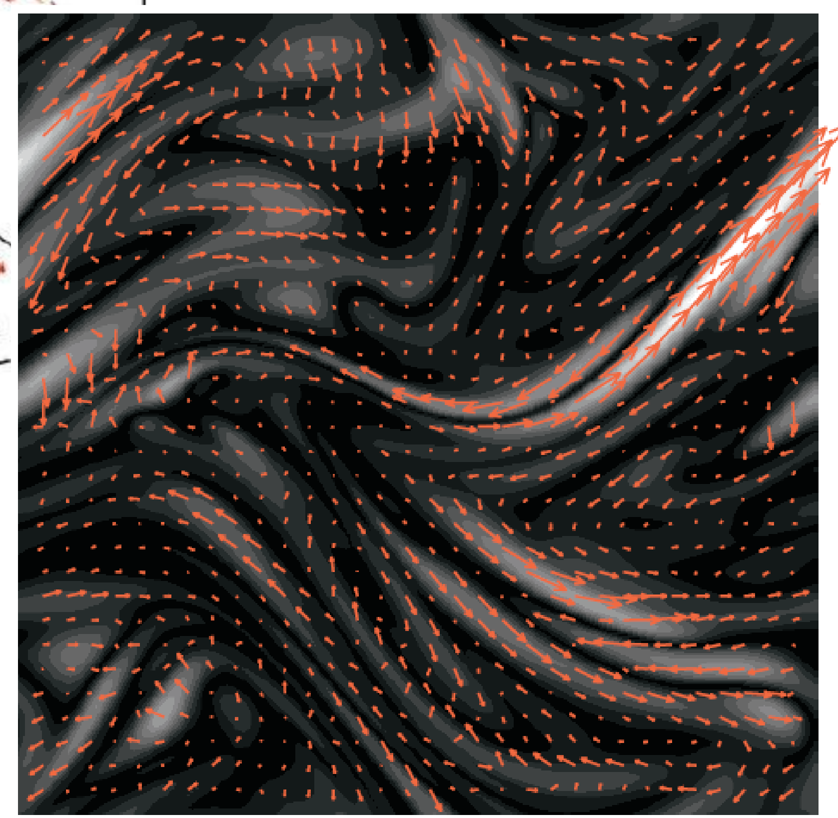
(3) a fast dynamo, $\gamma \simeq v_0/l_0$.

The fluctuation dynamo produces magnetic filaments and ribbons:

- length (radius of curvature) of order l_0,
- thickness of order $l_0 R_{\mathrm{m,cr}}^{-1/2} \simeq l_0/10$.

Haugen et al., PRE 2004



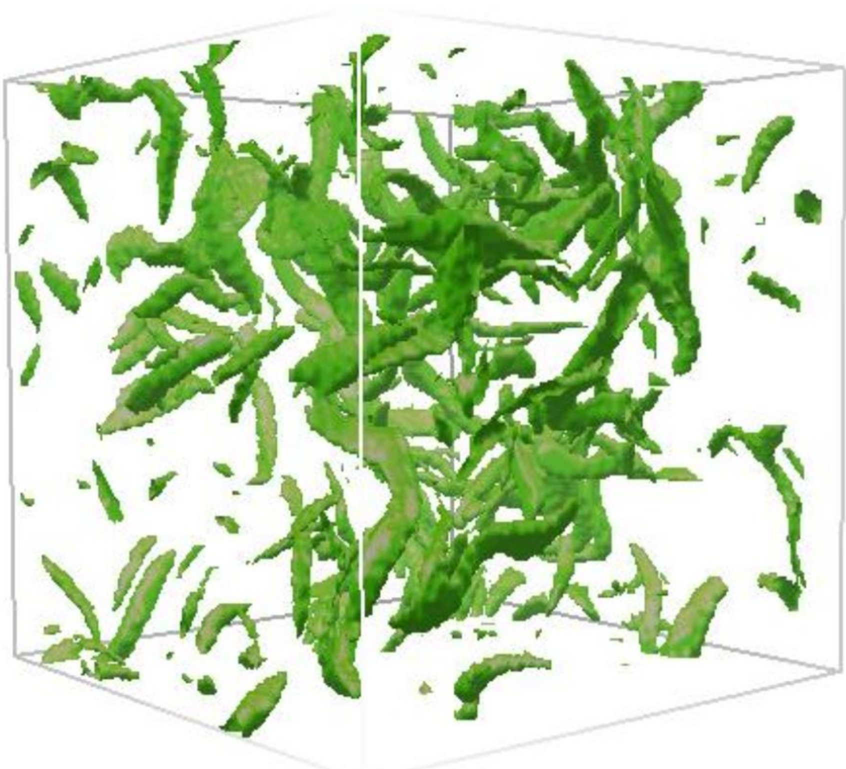


Schekochihin et al., ApJ 2004

Morphology of the magnetic structures

Wilkin et al., *PRL*, 2007: dynamo in an isotropic chaotic flow:

$$\vec{v} = \sum_{n=1}^{N} \left[\vec{C}_n \times \vec{k}_n \cos(\vec{k}_n \cdot \vec{x}) + \vec{D}_n \times \vec{k}_n \sin(\vec{k}_n \cdot \vec{x}) \right], \qquad E(k_n) \propto k_n^{-s}$$



Magnetic structures:

thickness: $l_1 \propto l_0 R_{\mathrm{m}}^{-2/(1-s)}$

width: $l_2 \propto l_0 R_{\mathrm{m}}^{-0.55}$

length: $l_3 \simeq l_0$

$R_{\mathrm{m}} \gg 1$: filaments, l_1 thick, l_0 long

Subramanian, *PRL*, 1999:

steady state: $R_{\mathrm{m,eff}} = R_{\mathrm{m,cr}} \simeq 10^2$

6.5. The mean-field dynamo

- Generates magnetic fields at a scale much larger than l_0.
- A fast dynamo.
- Requires that the random flow has broken mirror symmetry.
- The mirror symmetry breaks spontaneously due to rotation and stratification.

Notation from now on:

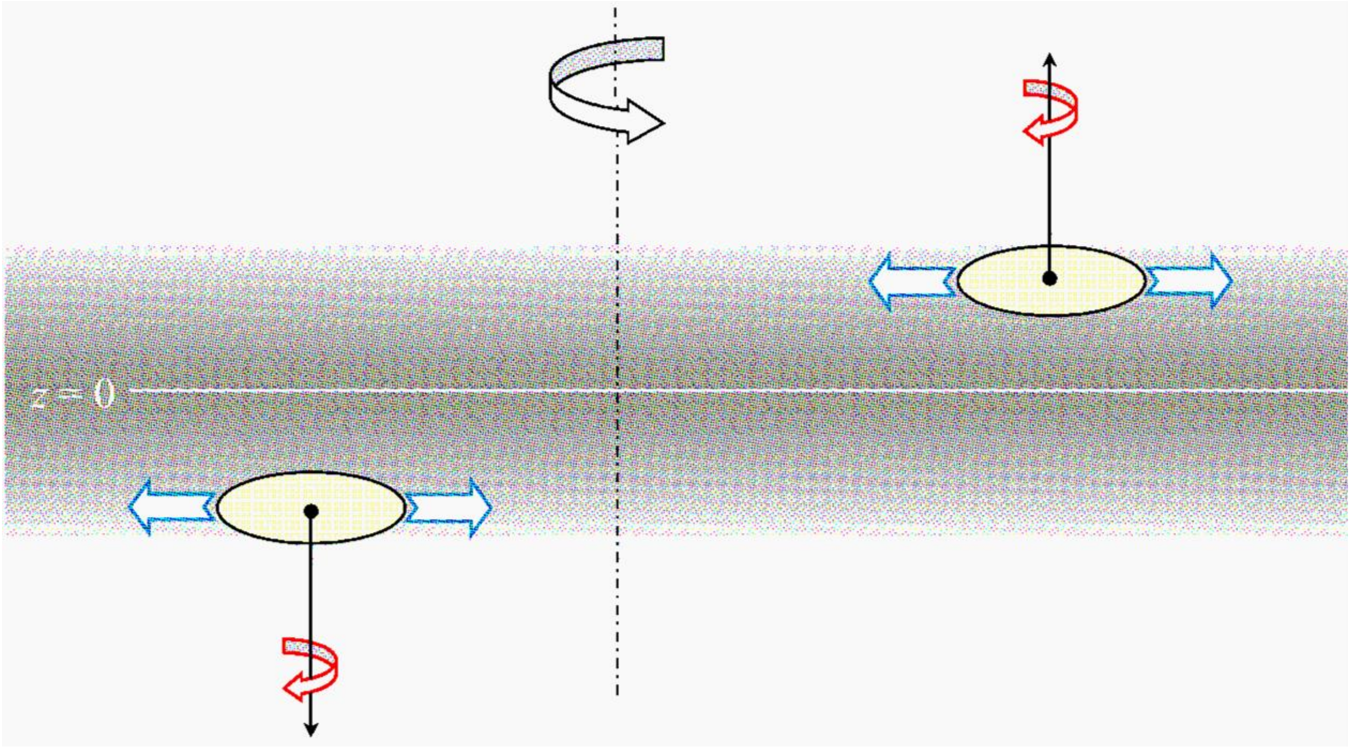
$\vec{B}$ = large-scale magnetic field,

$\beta \simeq l_0 v_0 / 3$ =turbulent magnetic diffusivity,

$\vec{V}$ = large-scale velocity field.

(A) Broken mirror symmetry (helicity) of interstellar turbulence:

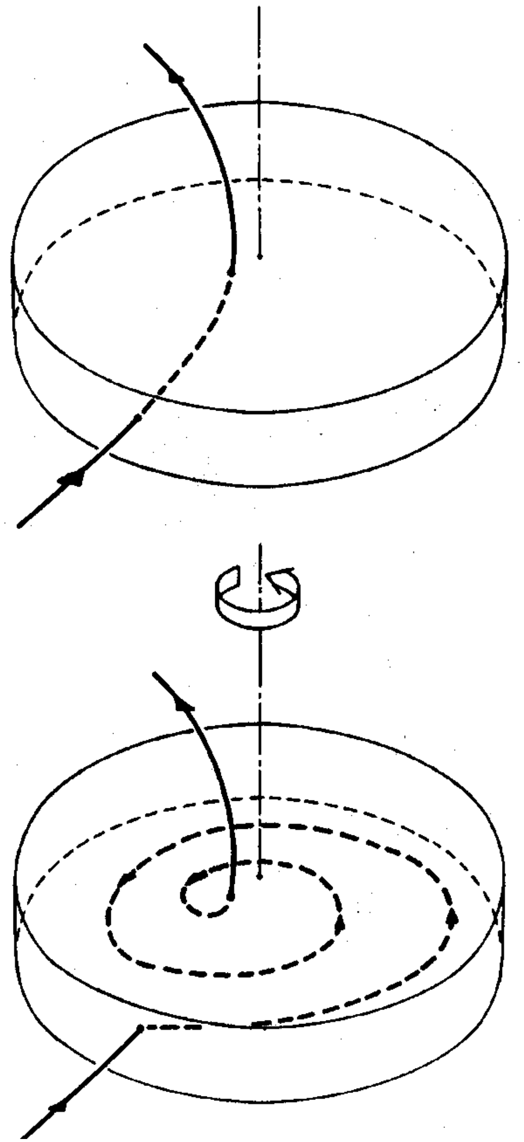
a consequence of angular momentum conservation in a rotating, stratified layer



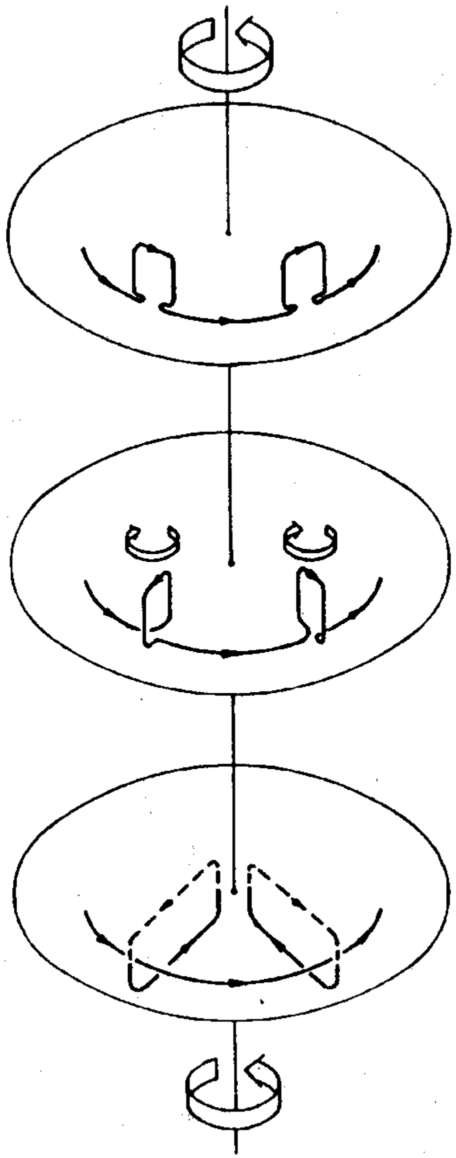
The helical fraction of the turbulent speed:

$$\alpha = -\tfrac{1}{3}\tau_0\langle\vec{v}\cdot\nabla\times\vec{v}\rangle \simeq \frac{l_0^2\Omega}{h} \simeq 1\,\mathrm{km/s}$$ (F. Krause, 1967).

(B) Differential rotation:
B_ϕ produced from B_r



(C) Helical turbulence:
B_r produced from B_ϕ



6.5.1. Basic equations

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\alpha \vec{B} + \vec{V} \times \vec{B}) + \beta \nabla^2 \vec{B}$$

$$\mathbf{B}(t, \mathbf{r}) = \tilde{\mathbf{B}}(t, r, z) e^{im\phi} \ , \qquad \epsilon = \frac{h_0}{R_0} \ll 1 \ .$$

Galactic discs are thin

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+imR_\omega\Omega+\frac{\epsilon^2m^2}{r^2}\right)\tilde{B}_r &= -R_\alpha\frac{\partial}{\partial z}(\alpha\tilde{B}_\phi)+\frac{\partial^2\tilde{B}_r}{\partial z^2}+\epsilon^2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{B}_r)\right] \\
&\quad + im\epsilon R_\alpha\frac{\alpha}{r}\tilde{B}_z-\frac{2im\epsilon^2}{r^2}\tilde{B}_\phi\ , \\
\left(\frac{\partial}{\partial t}+imR_\omega\Omega+\frac{\epsilon^2m^2}{r^2}\right)\tilde{B}_\phi &= R_\omega G\tilde{B}_r+R_\alpha\frac{\partial}{\partial z}(\alpha\tilde{B}_r)+\frac{\partial^2\tilde{B}_\phi}{\partial z^2} \\
&\quad +\epsilon^2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{B}_\phi)\right]-\epsilon R_\alpha\frac{\partial}{\partial r}(\alpha\tilde{B}_z) \\
&\quad +\frac{2im\epsilon^2}{r^2}\tilde{B}_r\ , \\
\left(\frac{\partial}{\partial t}+imR_\omega\Omega+\frac{\epsilon^2m^2}{r^2}\right)\tilde{B}_z &= \frac{\partial^2\tilde{B}_z}{\partial z^2}+R_\alpha\frac{\epsilon}{r}\frac{\partial}{\partial r}(r\alpha\tilde{B}_\phi)-im\epsilon R_\alpha\frac{\alpha}{r}\tilde{B}_r \\
&\quad +\epsilon^2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{B}_z)\right]+\frac{\epsilon^2}{r^2}\tilde{B}_z\ ,
\end{aligned}
$$

$$
\mathbf{B}(t,\mathbf{r})=\tilde{\mathbf{B}}(t,r,z)e^{im\phi}\ , \qquad \epsilon=\frac{h_0}{R_0}\ll 1\ , \qquad G=r\frac{d}{dr}\ .
$$

Thin disc, axisymmetric solutions, $\alpha^2\omega$-dynamo:

$$
\begin{aligned}
\frac{\partial B_r}{\partial t} &= -\frac{\partial}{\partial z}(\alpha B_\phi) + \beta \frac{\partial^2 B_r}{\partial z^2} \, , \\
\frac{\partial B_\phi}{\partial t} &= G B_r + \frac{\partial}{\partial z}(\alpha B_r) + \beta \frac{\partial^2 B_\phi}{\partial z^2} \, , \\
\frac{\partial B_z}{\partial t} &= \beta \frac{\partial^2 B_z}{\partial z^2} \, .
\end{aligned}
$$

$G = r\, d\Omega/dr$

Equation for B_z splits from the system.

B_z is supported through B_r and B_ϕ via $\partial/\partial r$

Dimensionless variables

$$\tilde{z} = \frac{z}{h} \Rightarrow \frac{\partial}{\partial z} = \frac{1}{h}\,\frac{\partial}{\partial \tilde{z}}\,, \qquad \tilde{t} = \frac{t}{h^2/\beta} \Rightarrow \frac{\partial}{\partial t} = \frac{\beta}{h^2}\,\frac{\partial}{\partial \tilde{t}}\,,$$

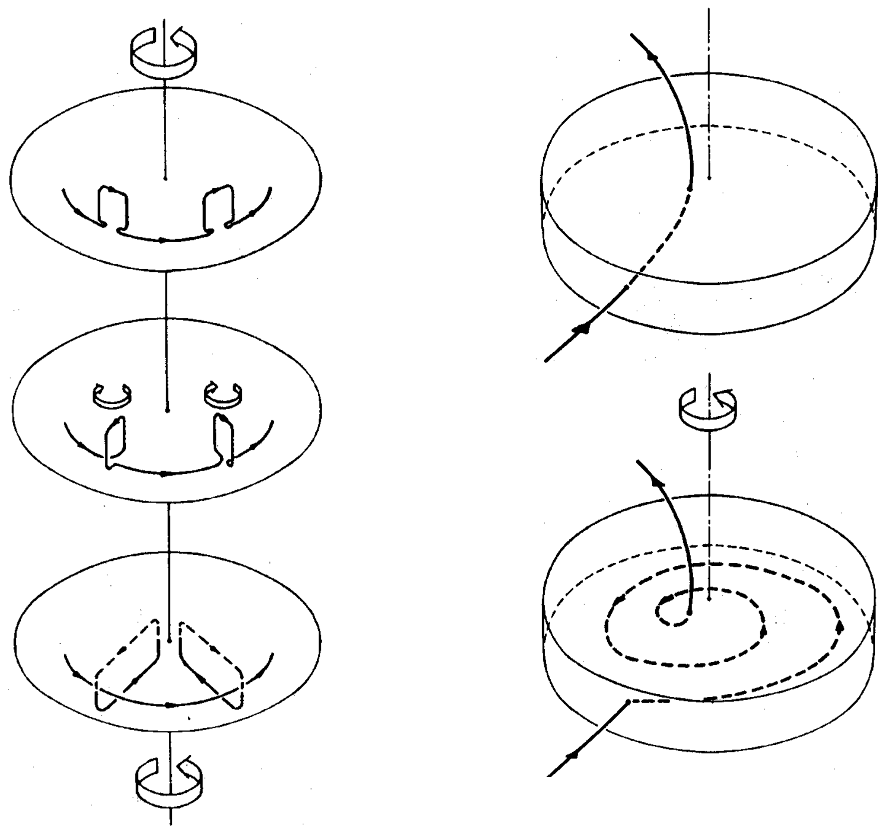
$$\tilde{\alpha} = \frac{\alpha(z)}{\alpha_0}\,.$$

$$\begin{aligned}
\frac{\partial B_r}{\partial \tilde{t}} &= -R_\alpha \frac{\partial}{\partial \tilde{z}}(\tilde{\alpha} B_\phi) + \frac{\partial^2 B_r}{\partial \tilde{z}^2}\,, \qquad R_\alpha = \frac{\alpha_0 h}{\beta} \\
\frac{\partial B_\phi}{\partial \tilde{t}} &= R_\omega B_r + R_\alpha \frac{\partial}{\partial \tilde{z}}(\tilde{\alpha} B_r) + \frac{\partial^2 B_\phi}{\partial \tilde{z}^2}\,, \qquad R_\omega = \frac{G h^2}{\beta}\,.
\end{aligned}$$

Drop ˜at dimensionless variables:

$$\frac{\partial B_r}{\partial t} = \underbrace{-R_\alpha \frac{\partial}{\partial z}(\alpha B_\phi)}_{B_\phi \to B_r \text{ via } \alpha\text{-effect}} + \frac{\partial^2 B_r}{\partial z^2},$$

$$\frac{\partial B_\phi}{\partial t} = \underbrace{R_\omega B_r}_{B_r \to B_\phi \text{ via differential rotation}} + \underbrace{R_\alpha \frac{\partial}{\partial z}(\alpha B_r)}_{B_r \to B_\phi \text{ via } \alpha\text{-effect}} + \frac{\partial^2 B_\phi}{\partial z^2}.$$



$\alpha\omega$-Dynamo: $|R_\omega| >> R_\alpha$

$$
\begin{aligned}
\frac{\partial B_r}{\partial t} &= -R_\alpha \frac{\partial}{\partial z}(\alpha B_\phi) + \frac{\partial^2 B_r}{\partial z^2} , \\
\frac{\partial B_\phi}{\partial t} &= R_\omega B_r + \frac{\partial^2 B_\phi}{\partial z^2} .
\end{aligned}
$$

Introduce new variable $B_r = R_\alpha B'_r$ and drop the dash:

$$
\begin{aligned}
\frac{\partial B_r}{\partial t} &= -\frac{\partial}{\partial z}(\alpha B_\phi) + \frac{\partial^2 B_r}{\partial z^2} , \\
\frac{\partial B_\phi}{\partial t} &= D B_r + \frac{\partial^2 B_\phi}{\partial z^2} ,
\end{aligned}
$$

where $D = R_\alpha R_\omega$ is the dynamo number.

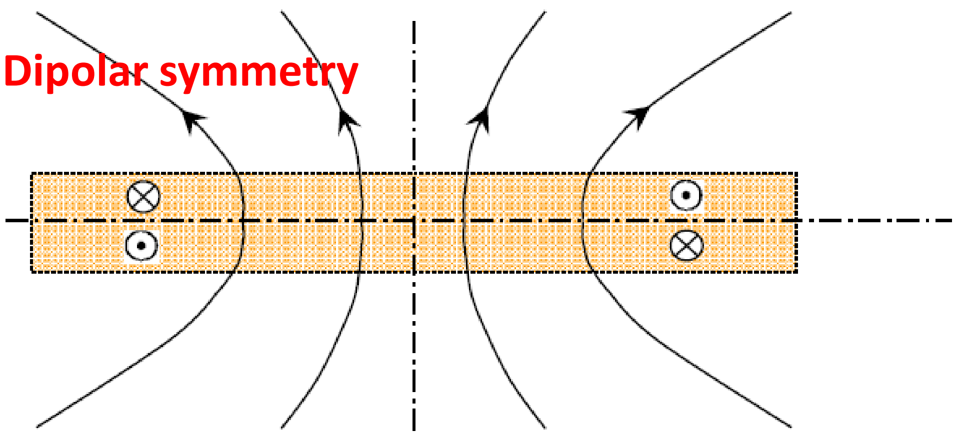
Boundary conditions

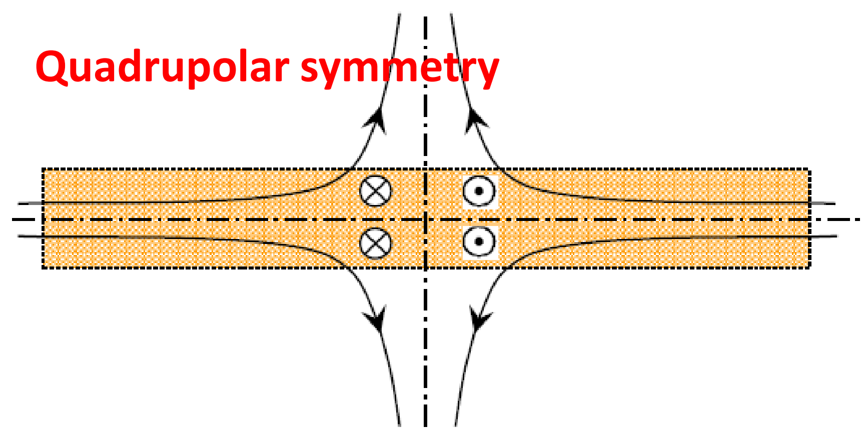
$$B_r\big|_{z=1} = B_\phi\big|_{z=1} = 0$$ (vacuum boundary conditions)

$$\left.\frac{\partial B_r}{\partial z}\right|_{z=0} = \left.\frac{\partial B_\phi}{\partial z}\right|_{z=0} = 0$$ (quadrupole)

$$B_r\big|_{z=0} = B_r\big|_{z=0} = 0$$ (dipole)

ϕ





6.5.2. Dynamo control parameters

NB! The Solar neighbourhood of the Milky Way, where these estimates apply, is not a typical galactic location.

Rotation $\quad = \dfrac{V_0}{r},$

$V_0 \simeq 200\,\mathrm{km/s}, \;\; r \simeq 10\,\mathrm{kpc},$

ionised gas scale height
$h \simeq 0.5\,\mathrm{kpc},$

turbulent velocity $v_0 \simeq 10\,\mathrm{km/s},$

turbulent scale $l_0 \simeq 0.1\,\mathrm{kpc}.$

$$\alpha_0 \simeq \frac{l_0^2}{h} \simeq 0.4\,\mathrm{km/s},$$

$$\beta \simeq \tfrac{1}{3} l_0 v_0 \simeq 10^{26}\,\mathrm{cm^2/s},$$

$$R_\alpha = \frac{\alpha_0 h}{\beta} \simeq 0.6$$

$$R_\omega = \frac{(r\,d\quad/dr)h^2}{\beta} \simeq -15$$

$$D = R_\alpha R_\omega \simeq -\left(\frac{3\quad h}{v_0}\right)^2 \simeq -10$$

6.5.3. The "no-z" approximation (Subramanian & Mestel, 1993)

Thin disc, dimensional $\alpha\omega$-dynamo equations:

$$\frac{\partial B_r}{\partial t} = -\frac{\partial}{\partial z}(\alpha B_\phi) + \beta \frac{\partial^2 B_r}{\partial z^2},$$

$$\frac{\partial B_\phi}{\partial t} = G B_r + \beta \frac{\partial^2 B_\phi}{\partial z^2}.$$

For solutions of a simple form, e.g., $B_{r,\phi} \propto \cos(\pi z/2h)$,

$$\frac{\partial}{\partial z} \simeq \frac{1}{h}, \qquad \frac{\partial^2}{\partial z^2} \simeq -\frac{1}{h^2}.$$

Kinematic solutions: $\vec{B} = \vec{B}_0 \exp(\gamma t)$.

$$\left(\gamma + \frac{\beta}{h^2}\right) B_{0r} + \frac{\alpha}{h} B_{0\phi} = 0,$$
$$-G B_{0r} + \left(\gamma + \frac{\beta}{h^2}\right) B_{0\phi} = 0.$$

Nontrivial solutions exist if

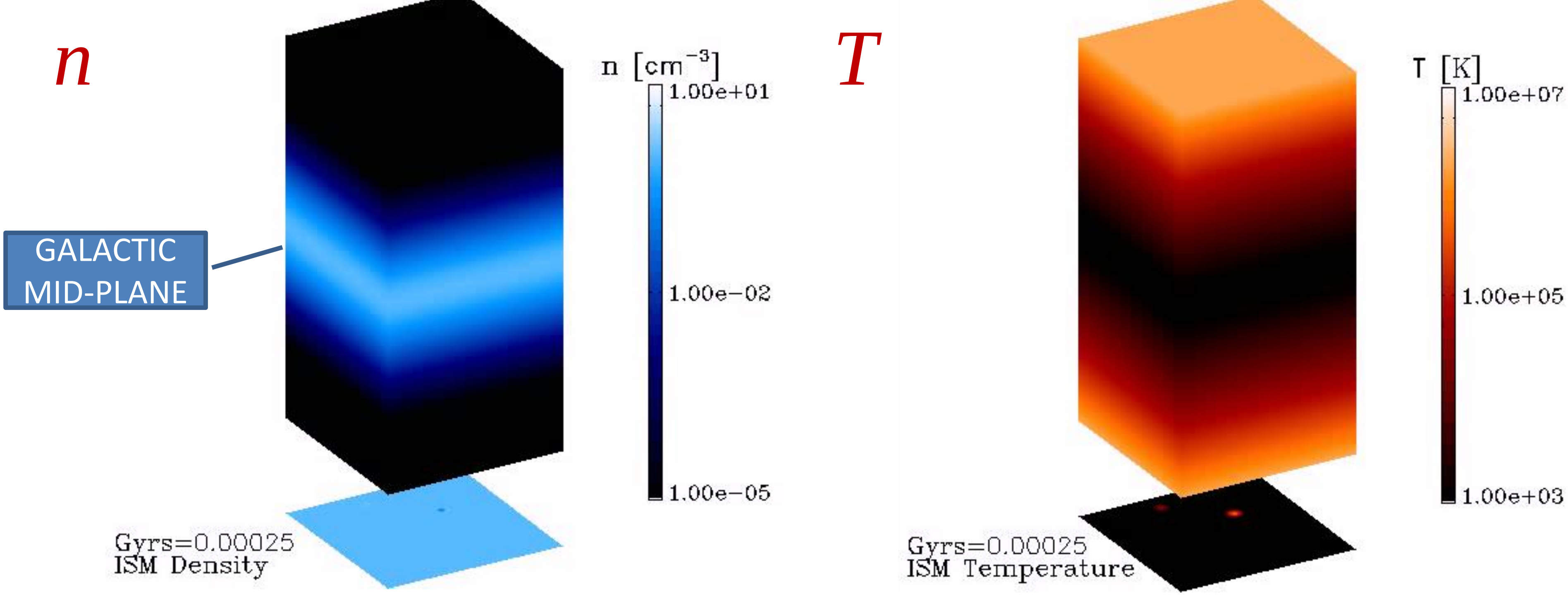
$$\begin{vmatrix} \gamma + \beta/h^2 & \alpha/h \\ -G & \gamma + \beta/h^2 \end{vmatrix} = 0,$$

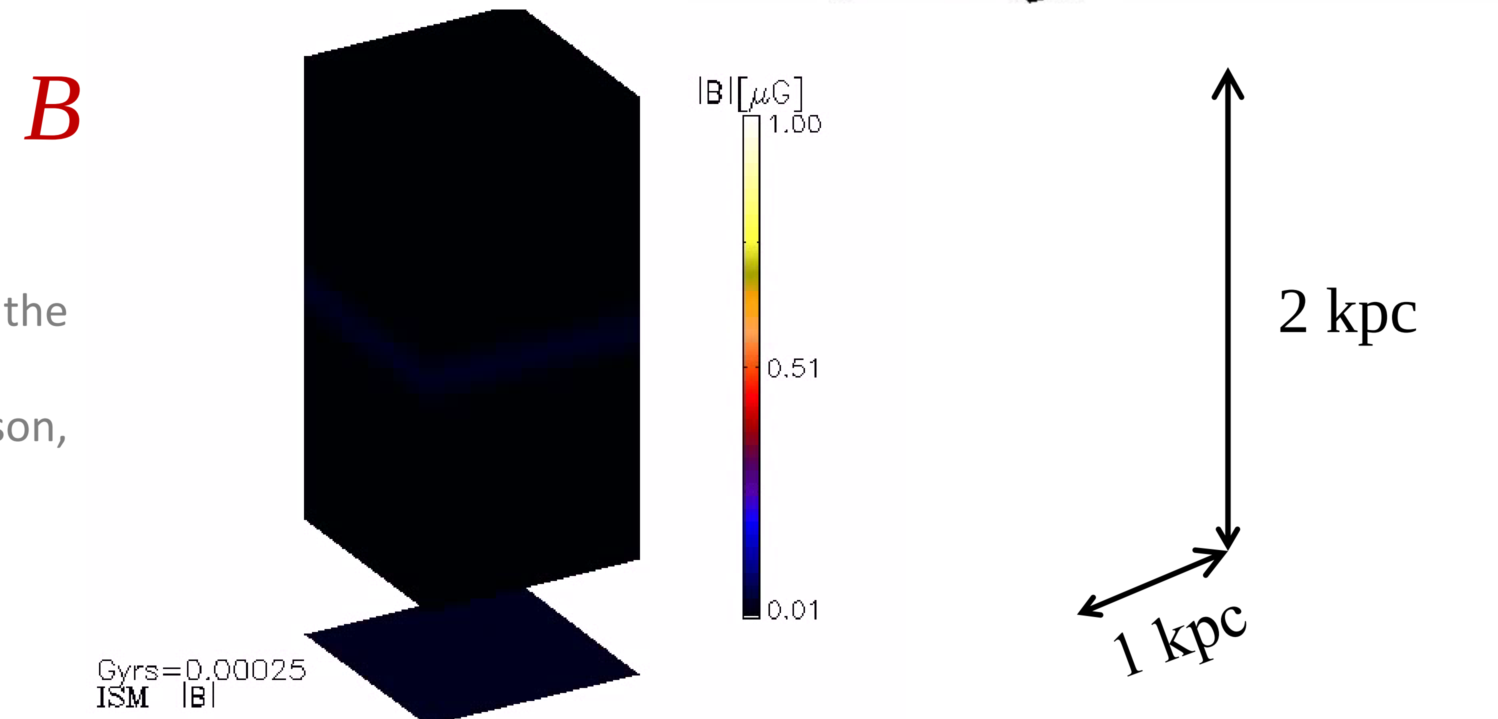
i.e., $\gamma \simeq \frac{\beta}{h^2}(-1 + \sqrt{-D}),$

$$\tan p = \frac{B_r}{B_\phi} \simeq -\sqrt{\frac{\alpha}{-Gh}} = -\sqrt{\frac{R_\alpha}{|R_\omega|}}.$$

Magnetic field grows if $D \lesssim -1$, with $p \simeq -\arctan \frac{1}{4} \simeq -15°$.

NB! Magnetic pitch angle in nonlinear solutions is generally smaller, so the agreement between the linear solutions and observations may be a coincidence!





MHD simulations of the SN-driven ISM:
Gent, Shukurov, Sarson, Fletcher & Mantere, 2013, MNRAS