Diffusive Shock Acceleration and magnetic fields in SNR

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SN1006: A supernova remnant 7,000 light years from Earth
Cosmic ray spectrum arriving at earth

Mainly protons
CR populations

(I) Galactic Supernova remnants

(II) Probably galactic

(III) Extra-galactic ?

\[ \log(\text{FLUX} \times E^3) \text{ in eV}^2\text{m}^{-2}\text{s}^{-1}\text{Sr}^{-1} \]

\[ \log(\text{ENERGY in eV}) \]

knee \[ \sim E^{-2.6} \]

ankle \[ \sim E^{-3} \]
Why shocks?
Historical shell supernova remnants

Tycho 1572AD

Kepler 1604AD

SN1006

Cas A 1680AD

Chandra observations

NASA/CXC/Rutgers/ J.Hughes et al.

NASA/CXC/Rutgers/ J.Warren & J.Hughes et al.

NASA/CXC/NCSU/ S.Reynolds et al.

NASA/CXC/MIT/UMass Amherst/ M.D.Stage et al.
Cassiopeia A

Radio (VLA)
Infrared (Spitzer)
Optical (Hubble)
X-ray (Chandra)

Mixture of line radiation & synchrotron continuum

Synchrotron in magnetic field ~ 0.1-1mG
Radio ($h\nu \sim 10^{-5}$eV): electron energy ~1 GeV
X-ray ($h\nu \sim 10^{3}$eV): electron energy ~ 10 TeV
HESS: \( \gamma \)-rays directly produced by TeV particles

SNR RX J1713.7-3946

Aharonian et al
Active galaxies

Centaurus A is the closest powerful radio galaxy (5Mpc)

optical

Radio jets

X-ray (Chandra)
Strong shock: high Mach number

Conserved across shock (Rankine Hugoniot relations)

- **Mass flux**: \( \rho u \)
- **Momentum flux**: \( P + \rho u^2 \)
- **Energy flux**: \( \frac{5}{2} Pu + \frac{1}{2} \rho u^3 \)

In shock rest frame

- \( u = u_{shock} \)
- \( \rho = \rho_0 \)
- \( kT = 0 \)
- \( P = 0 \)

Downstream

- \( u = \frac{1}{4} u_{shock} \)
- \( \rho = 4 \rho_0 \)
- \( kT = \frac{3}{16} \left( \frac{\overline{A}}{1 + \overline{Z}} \right) m_p u_{shock}^2 \)
- \( P = \frac{3}{4} \rho u_{shock}^2 \)

Shock turns kinetic streaming energy into random thermal energy

Divert part of thermal energy into high energy particles
Cosmic ray acceleration by shocks
Cosmic ray acceleration

Due to scattering, CR recrosses shock many times
Gains energy at each crossing
Shock acceleration energy spectrum: energy gain

Change in fluid velocity across shock
\[
\Delta v = v_s - \frac{v_s}{4} = \frac{3v_s}{4}
\]

Change in momentum from upstream to downstream
\[
\Delta p_1 = p' - p = p \frac{\Delta v}{c} \cos \vartheta
\]

Mean increase in momentum
\[
< \Delta p_1 > = \frac{\int_0^{\pi/2} \Delta p_1 \cos \vartheta \sin \vartheta d\vartheta}{\int_0^{\pi/2} \cos \vartheta \sin \vartheta d\vartheta} = (\frac{v_s}{2c})p
\]

Similar increase in momentum on recrossing into upstream
\[
< \Delta p_2 > = (\frac{v_s}{2c})p
\]

Average fractional energy gained at each crossing is
\[
\frac{\Delta \varepsilon}{\varepsilon} = \frac{< \Delta p_1 > + < \Delta p_2 >}{p} = \frac{v_s}{c}
\]
Shock acceleration energy spectrum: loss rate

CR cross from upstream to downstream at rate \( nc/4 \)

CR carried away downstream at rate \( n \nu_{\text{downstream}} = n \nu_s/4 \)

Mean number of shock crossings = \( (nc/4)/(n \nu_s/4) = c/\nu_s \)

Fraction lost at each shock crossing is \( \nu_s/c \)

\[
\frac{\Delta n}{n} = -\frac{\nu_s}{c}
\]
**Shock acceleration energy spectrum**

Fractional CR loss per shock crossing

\[ \frac{\Delta n}{n} = - \frac{v_s}{c} \]

Fractional energy gain per shock crossing

\[ \frac{\Delta \varepsilon}{\varepsilon} = \frac{v_s}{c} \]

Turn into differential equation

\[ \frac{dn}{d\varepsilon} \approx \frac{\Delta n}{\Delta \varepsilon} = -\frac{n}{\varepsilon} \quad \Rightarrow \quad n \propto \varepsilon^{-1} \]

integrated spectrum

Differential energy spectrum

\[ N(\varepsilon) d\varepsilon \propto \varepsilon^{-2} d\varepsilon \]
Derivation from Boltzmann equation

Krimskii 1977
Axford, Leer & Skadron 1977
Blandford & Ostriker 1978
The Vlasov-Fokker-Planck (VFP) equation

\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{p}} = C(f) \]

Vlasov equation (collisionless)

Collisions Fokker-Planck

\[ f(x, y, z, p_x, p_y, p_z, t) \, dx \, dy \, dz \, dp_x \, dp_y \, dp_z \]

= number of CR in phase space volume \( dx \, dy \, dz \, dp_x \, dp_y \, dp_z \)

VFP equation:

1) Advection: at velocity \( \mathbf{v} \) in \( r \)-space

   at velocity \( e(\mathbf{E}+\mathbf{v} \times \mathbf{B}) \) in \( p \)-space

   B on scale > CR Larmor radius

2) Collisions: small angle scattering

   Due to B on scale < CR Larmor radius
Cosmic ray acceleration

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - e(E + v \times B) \cdot \frac{\partial f}{\partial p} = C(f) \]
Parallel shock

\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - e(E + v \times B) \cdot \frac{\partial f}{\partial p} = C(f) \]

Only diffusion along $B$ matters

Large scale field irrelevant

Same is if no large scale field
Redefine $f$ in local fluid rest frame

$f$ fluid rest frame

Fluid moves at velocity $u$

$$\frac{\partial f}{\partial t} + (v_x + u) \frac{\partial f}{\partial x} - \frac{\partial u}{\partial x} p_x \frac{\partial f}{\partial p_x} = C(f)$$

advection with fluid

Frame transformation
Sub-relativistic shocks: small $u/c$

To first approximation in $u/c$

\[ f = f_0(p) + f_1(p) \frac{p_x}{p} \]

Isotropic

Drift

VFP equation reduces to

\[
\frac{\partial f_0}{\partial t} + u \frac{\partial f_0}{\partial x} + \frac{v}{3} \frac{\partial f_1}{\partial x} - \frac{1}{3} \frac{\partial u}{\partial x} p \frac{\partial f_0}{\partial p} = 0
\]

Advection

Diffusion

Adiabatic compression

\[
\frac{\partial f_0}{\partial t} + u \frac{\partial f_0}{\partial x} - \frac{\partial v}{\partial x} \left( \frac{v v}{3} \frac{\partial f_0}{\partial x} \right) - \frac{1}{3} \frac{\partial u}{\partial x} p \frac{\partial f_0}{\partial p} = 0
\]

Scattering frequency

\[ v \frac{\partial f_0}{\partial x} = -v f_1 \]
Steady state solution

\[ f = f_0(p) + f_1(p) \frac{p_x}{p} \]

\[ u_1 = u_{\text{shock}} \quad \text{u}_2 = u_{\text{shock}}/4 \]

No escape upstream: \[ u_1 f_0 + \frac{c}{3} f_1 = 0 \]

Downstream: no drift relative to background \[ f_1 = 0 \]

Boundary condition at shock

\[ \left[ f_1 - \frac{u_1}{c} p \frac{\partial f_0}{\partial p} \right]_{\text{upstream}} = \left[ f_1 - \frac{u_2}{c} p \frac{\partial f_0}{\partial p} \right]_{\text{downstream}} \]

\[ (u_1 - u_2) p \frac{\partial f_0}{\partial p} = -3u_1 f_0 \]

\[ f_0 \propto p^{-4} \]
Acceleration efficiency
Efficiency

- Has to be efficient (10-50%) to explain galactic CR energy density

- Solar wind shocks can be >10% efficient

- Shock processes produce many suprathermal protons
At high efficiency: non-linear feedback onto shock

Drury & Voelk (1981)

CR pressure decelerates flow into shock
High efficiency: concave spectrum

Steep spectrum

Flat spectrum

SN1006 (Allen et al 2008)
Maximum CR energy
CR upstream of shock

Balance between:
- flow into shock
- diffusion away from shock

\[ \frac{\partial n_{cr}}{\partial t} = -u \frac{\partial n_{cr}}{\partial x} - D \frac{\partial^2 n_{cr}}{\partial x^2} = 0 \]

Exponential density
\[ n_{cr} = n_0 e^{ux/D} \]

Scaleheight
\[ L = \frac{D}{u} \]
CR acceleration time

Number of CR upstream:  \( n_{cr} \frac{D_{\text{upstream}}}{u_{\text{shock}}} \)

Rate CR cross shock:  \( \frac{1}{4} n_{cr} c \)

Average time spent upstream:  \( \Delta t = \frac{4D_{\text{upstream}}}{cu_{\text{shock}}} \)  (neglect time spent downstream)

Average energy gain per shock crossing:  \( \frac{\Delta \varepsilon}{\varepsilon} = \frac{u_{\text{shock}}}{c} \)

Acceleration rate:  \( \frac{\Delta \varepsilon}{\Delta t} = \frac{\varepsilon}{4} \frac{u_{\text{shock}}^2}{D_{\text{upstream}}} \)

Time needed for acceleration (Lagage & Cesarsky)  \( \tau = \frac{4D_{\text{upstream}}}{u_{\text{shock}}^2} + \frac{4D_{\text{downstream}}}{(u_{\text{shock}}/4)^2} \)  downstream time
**Maximum CR energy**

Acceleration time
\[ \tau = \frac{8D}{u_{shock}^2} \]

Diffusion coefficient
\[ D = \frac{\lambda c}{3} \]

\[ \lambda \leq \frac{3}{8} \frac{u_{shock}}{c} R \]

where \( R = u_{shock} \tau \)

Smallest possible mfp:
\[ \lambda = \frac{p_{cr}}{eB} \]

Limit on CR momentum:
\[ p_{cr} = \frac{3}{8} \frac{u_{shock}}{c} eBR \]

Limit on CR energy in eV:
\[ \frac{cp_{cr}}{e} = \frac{3}{8} Bu_{shock} R \]

Typically for young SNR

ISM mag field: few \( \mu \)G

\[ u_{shock} = \frac{c}{30} \]

Max CR energy \( \sim 10^{14} \)eV

Under favourable assumptions
Figure 1  Size and magnetic field strength of possible sites of particle acceleration. Objects below the diagonal line cannot accelerate protons to $10^{20}$ eV.
Perpendicular shocks

(Jokipii 1982, 1987)
CR trajectory at perpendicular shock (no scattering)

\[ E = -u_{\text{shock}} \times B \]

\[ v_{\text{drift}} = \frac{E \times B}{B^2} \]

CR gain energy by drifting in E field
CR acceleration at perpendicular shock

CR trajectory divides into

- Motion of gyrocentre
- Gyration about gyrocentre

Without diffusion:
Every CR gets small adiabatic gain due to compression at shock
CR acceleration at perpendicular shock: with scattering

Diffusive shock theory applies
Provided gyrocentre diffuses over distances greater than Larmor radius during shock transit
Same power law (see later)

No scattering
Weak scattering
Strong scattering

Not to scale
CR acceleration at perpendicular shock

\[ E = -u_{shock} \times B \]

Transit between pole & equator: energy gain \( \sim eER = eu_{shock}BR \)

Hillas parameter as with parallel shock: similar max CR energy
The case of SN1006

Polar x-ray synchrotron emission?
(Rothenflug et al 2004)

At perpendicular shocks

• Acceleration is faster – potentially higher CR energy

• CR energy limited to $euBR$ (Hillas) by space rather than time

• Injection is more difficult at a perpendicular shock

• CR scattering frequency has to be in right range

Room for discussion!
CR scattering

what is the mean free path?
CR drive a ‘resonant’ instability

Spatial resonance between wavelength and CR Larmor radius

wave deflects CR ↔ CR current drives wave

Skilling (1975)

Wave growth (energy density $I$)

$$\frac{\partial I}{\partial t} + u \frac{\partial I}{\partial x} = \frac{v_A p c}{U_M} \frac{\partial n_{cr}}{\partial x}$$

$$I = \frac{\delta B^2}{B^2}$$

CR scattering

$$\frac{\partial n_{cr}}{\partial t} + u \frac{\partial n_{cr}}{\partial x} = \frac{\partial}{\partial x} \left( D \frac{\partial n_{cr}}{\partial x} \right)$$

$$D = \frac{4}{3\pi} \frac{c r_g}{I}$$
Turbulence upstream of shock

Skilling (1975)

Wave growth (amplitude $I$)

$$\frac{\partial I}{\partial t} + u \frac{\partial I}{\partial x} = \frac{v_A p c}{U_M} \frac{\partial n_{cr}}{\partial x}$$

CR scattering

$$\frac{\partial n_{cr}}{\partial t} + u \frac{\partial n_{cr}}{\partial x} = \frac{\partial}{\partial x} \left( D \frac{\partial n_{cr}}{\partial x} \right)$$

$$D = \frac{4 \frac{c r_g}{3\pi} I}{I}$$

Solution

$$I_{\text{shock}} = M_A \frac{U_{cr}}{\rho u_{\text{shock}}}$$

$$\text{mfp} = \frac{4 \frac{r_g}{\pi} I}{I}$$

$$L = \frac{4 \frac{c r_g}{3 u_{\text{shock}}} I}{I}$$

Alfven Mach number ~1000

CR efficiency ~0.1

Question: What does $I > 1$ tell us?

?Implies?: $\text{mfp} < \text{Larmor radius}$

Waves non-linear: $I >> 1$
Evidence that magnetic field exceeds typical interstellar value

1) x-ray observations

2) Needed to accelerate galactic CR to a few PeV

3) Theory breaks down because of large wave growth

\[ I_{\text{shock}} = M_A \frac{U_{cr}}{\frac{\rho u_{\text{shock}}}{2}} \]

Alfven Mach number \(\sim 1000\)  
CR efficiency \(\sim 0.1\)
Evidence that magnetic field exceeds typical interstellar value

1) x-ray observations

2) Needed to accelerate galactic CR to a few PeV

3) Theory breaks down because of large wave growth

\[ I_{shock} = M_A \frac{U_{cr}}{\rho u_{shock}^2} \]

Alfven Mach number \( \sim 1000 \)

CR efficiency \( \sim 0.1 \)

Need to look more closely at CR interaction with magnetic field
Diffusive Shock Acceleration and magnetic fields in SNR

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SN1006: A supernova remnant 7,000 light years from Earth
magnetic field amplification
and
cosmic ray scattering
Streaming CR excite instabilities

CR streaming ahead of shock
Excite instabilities
Amplify magnetic field
Streaming instabilities amplify magnetic field
Lucek & Bell (2000)

B field lines, t = 0

CR treated as particles

Thermal plasma as MHD
Electric currents carried by CR and thermal plasma

Density of $10^{15}$eV CR: $\sim 10^{-12}$ cm$^{-3}$
Current density: $j_{cr} \sim 10^{-18}$ Amp m$^{-2}$

CR current must be balanced by current carried by thermal plasma

$$j_{thermal} = -j_{cr}$$

$j_{thermal} \times B$ force acts on plasma to balance $j_{cr} \times B$ force on CR
Three equations control the instability

1) \[ \rho \frac{du}{dt} = -\nabla p - \frac{1}{\mu_0} B \wedge (\nabla \wedge B) - j_{cr} \wedge B \]

2) \[ \frac{\partial B}{\partial t} = \nabla \wedge (u \wedge B) \]

3) Equation for \( j_{cr} \) in terms of perturbed \( B \)

\[ j \times B \text{ driving force splits into two parts:} \]

\[ \rho \frac{du}{dt} = -\nabla p - \frac{1}{\mu_0} B \times (\nabla \times B) - j_{cr \perp} \times B - j_{cr \parallel} \times B \]
Resonant Alfvén instability

\[ \rho \frac{\partial u}{\partial t} = -\nabla p - \frac{1}{\mu_0} B \times (\nabla \times B) - j_{cr \perp} \times B - j_{cr \parallel} \times B \]

\[ j_{cr \perp} \times B \] drives Alfvén waves

Perturbed cosmic ray current
Non-resonant instability

\[ \rho \frac{\partial u}{\partial t} = -\nabla p - \frac{1}{\mu_0} B \times (\nabla \times B) - j_{cr\perp} \times B - j_{cr\parallel} \times B \]

\[ j_{cr\parallel} \times B \] dominates for shock acceleration in SNR

Perturbed magnetic field
Dispersion relation

Wavelength longer than Larmor radius
CR follow field lines.
jxB drives weak instability

Magnetic tension inhibits instability

\[ \rho \frac{du}{dt} = -j_{CR} \times B \]

\[ \frac{\partial B}{\partial t} = \nabla \times (u \times B) \]

\[ \gamma = \left( \frac{KB_0 j_{CR}}{\rho} \right)^{1/2} \]

k in units of \( r_g^{-1} \)
\( \omega \) in units of \( v_s^2/cr_g \)

Red line is growth rate
The essence of the non-resonant instability
Spiral expands leaving central cavity

Same without vertical field
Simplest form: expanding loops of $B$.

$j \times B$ expands loops

$\rightarrow$ stretches field lines

$\rightarrow$ more $B$

$\rightarrow$ more $j \times B$
Simplest form: expanding loops of $B$

$j \times B$ expands loops

$\rightarrow$ stretches field lines

$\rightarrow$ more $B$

$\rightarrow$ more $j \times B$

CR current
Non-linear growth – expanding loops

Slices through $|B|$ - time sequence (fixed CR current)

Field lines: wandering spirals

Cavities and walls in $|B|$ & $\rho$
How large does the magnetic field grow?
Historical SNR

Tycho 1572AD
Kepler 1604AD
SN1006
Cas A 1680AD

Chandra observations

NASA/CXC/Rutgers/
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M.D.Stage et al.
Instability growth

No reason for non-linear saturation of a single mode
Saturation (back of envelope)

Magnetic field grows until

1) \[ \frac{1}{\mu_0} B \times (\nabla \times B) \approx j_{CR} \times B \]

Magnetic tension            CR driving force

2) \[ \frac{p}{eB} \approx L \]

CR Larmor radius             scalelength

Set  \[ \nabla = \frac{1}{L} \]  and eliminate \( L \) between 1) & 2)

\[ \frac{B^2}{\mu_0} \approx \frac{pj_{CR}}{e} \approx \text{efficiency} \times \frac{\rho u_{shock}^3}{c} \]
Inferred downstream magnetic field (Vink 2008)

\[ B^2/(8\pi\rho) \text{ (cgs)} \]

\[ B \approx 700 \left( \frac{u}{10^4 \text{ kms}^{-1}} \right)^{3/2} \left( \frac{n_e}{\text{cm}^{-3}} \right)^{1/2} \mu\text{G} \]

Theory:

\[ B \approx 400 \left( \frac{u}{10^4 \text{ kms}^{-1}} \right)^{3/2} \left( \frac{n_e}{\text{cm}^{-3}} \right)^{1/2} \left( \frac{\eta}{0.1} \right)^{1/2} \mu\text{G} \]
The cosmic ray spectrum

revision from \( p^{-4} \)
Shock acceleration energy spectrum: loss rate

CR cross from upstream to downstream at rate \( n_{\text{shock}} c/4 \)

CR carried away downstream at rate = \( n_{\text{downstream}} v_s/4 \)

Fraction lost at each shock crossing:

\[
\frac{\Delta n}{n} = - \frac{v_s}{c} \frac{n_{\text{downstream}}}{n_{\text{shock}}}
\]

In diffusive limit, \( n_{\text{downstream}} = n_{\text{shock}} \)
Parallel shock

$u/c = 0.1$

$v/\omega_g = 0.1$
Vlasov-Fokker-Planck (VFP) analysis for oblique magnetic field
\[
\frac{\partial f}{\partial t} + (v_x + u) \frac{\partial f}{\partial x} - \frac{\partial u}{\partial x} p_x \frac{\partial f}{\partial p_x} + e \mathbf{v} \times \mathbf{B}, \frac{\partial f}{\partial p} = C(f)
\]

Equivalent form of solution for small \( u/c \)

\[
f = f_0(p) + f_x(p) \frac{p_x}{p} + f_y(p) \frac{p_y}{p} + f_z(p) \frac{p_z}{p}
\]

Upstream solution

\[
f_x = -\frac{3u_1}{c} f_0 \quad f_y = \frac{3u_1}{c} \frac{v \omega_z}{\sqrt{v^2 + \omega_x^2}} f_0 \quad f_z = \frac{3u_1}{c} \frac{\omega_z^2}{\sqrt{v^2 + \omega_x^2}} f_0
\]

Downstream solution

\[
f_x = f_y = f_z = 0
\]

\( \omega = \frac{e \mathbf{B}}{\gamma m_p} \)

Cannot match \( f_y \) & \( f_z \) across the shock
\[ \frac{\partial f}{\partial t} + (v_x + u) \frac{\partial f}{\partial x} - \frac{\partial u}{\partial x} p_x \frac{\partial f}{\partial p_x} + e v \times B \cdot \frac{\partial f}{\partial p} = C(f) \]

Equivalent form of solution for small u/c

\[ f = f_0(p) + f_x(p) \frac{p_x}{p} + f_y(p) \frac{p_y}{p} + f_z(p) \frac{p_z}{p} \]

Upstream solution

\[ f_x = -\frac{3u_1}{c} f_0 \]
\[ f_y = \frac{3u_1}{c} \frac{v \omega_z}{v^2 + \omega_x^2} f_0 \]
\[ f_z = \frac{3u_1}{c} \frac{\omega_z^2}{v^2 + \omega_x^2} f_0 \]

Downstream solution

\[ f_x = f_y = f_z = 0 \]

Cannot match \( f_y \) & \( f_z \) across the shock

\[ \omega = \frac{eB}{\gamma m_p} \]
Expand in spherical harmonics

$$f(x, p, t) = \sum_{l,m} f_l^m(x, |p|, t) P_l^m(\cos \theta) \exp(i m \phi)$$
Equation for evolution of each spherical harmonic

\[ f(x, p, t) = \sum_{l,m} f_l^m(x, p, t) P_l^m(\cos \theta) e^{im\phi} \]

\[
\frac{\partial f_l^m}{\partial t} + u \frac{\partial f_l^m}{\partial x} + c \left[ \frac{l - m}{2l - 1} \frac{\partial f_{l-1}^m}{\partial x} + \frac{l + m + 1}{2l + 3} \frac{\partial f_{l+1}^m}{\partial x} \right] \\
+ im \frac{ceB_x}{p} f_l^m + \frac{ceB_z}{2p} \beta_l^m \\
-p \frac{\partial u}{\partial x} \left[ \frac{(l - m)(l - m - 1)}{(2l - 3)(2l - 1)} \left( \frac{\partial f_{l-2}^m}{\partial p} - (l - 2) \frac{f_{l-2}^m}{p} \right) \\
+ \frac{(l - m)(l + m)}{(2l - 1)(2l + 1)} \left( \frac{\partial f_{l}^m}{\partial p} + (l + 1) \frac{f_{l}^m}{p} \right) \\
+ \frac{(l - m + 1)(l + m + 1)}{(2l + 1)(2l + 3)} \left( \frac{\partial f_{l+1}^m}{\partial p} - l \frac{f_{l+1}^m}{p} \right) \right] \\
+ \frac{(l + m + 1)(l + m + 2)}{(2l + 3)(2l + 5)} \left( \frac{\partial f_{l+2}^m}{\partial p} + (l + 3) \frac{f_{l+2}^m}{p} \right) \right] \\
-pu \frac{\partial u}{\partial x} \left[ \frac{l - m}{2l - 1} \left( \frac{\partial f_{l-1}^m}{\partial p} - (l - 1) \frac{f_{l-1}^m}{p} \right) \\
+ \frac{l + m + 1}{2l + 3} \left( \frac{\partial f_{l+1}^m}{\partial p} + (l + 2) \frac{f_{l+1}^m}{p} \right) \right] \\
= -\frac{l(l + 1)}{2} \nu f_l^m
\]

where \( \beta_l^m = (l - m)(l + m + 1) f_l^{m+1} - f_l^{m-1} \) for \( m > 0 \) and \( \beta_l^0 = 2\Re(f_l^1). \)
Parallel shock

\[ \theta = 0^\circ \]

\[ \frac{u}{c} = 0.1 \]

\[ \frac{v}{v_g} = 0.1 \]
Oblique shock (nearly parallel)

\[ \theta = 30^\circ \]

\[ \frac{u}{c} = 0.1 \]

\[ \frac{v}{\omega_g} = 0.1 \]

Density spike at shock as seen by
More perpendicular, less parallel

\[ \theta = 60^\circ \]

\[ u/c = 0.1 \quad \nu / \omega_g = 0.1 \]

\[ Re(f_1^1) \text{ represents cross-field drift} \]

\[ Im(f_1^1) \text{ represents drift along oblique field lines} \]
Peperpendicular shock

\[ \theta = 90^\circ \]

\[ u/c = 0.1 \]

\[ \nu/\omega_g = 0.1 \]

CR density reduced at shock
CR density profiles near shock

- $u/c = 0.1$
- $\nu/\omega_g = 0.1$

- $\theta = 90^\circ$
- $\theta = 72^\circ$
- $\theta = 60^\circ$

CR density

Distance from shock (CR Larmor radius)

Upstream downstream
Spectral index plotted against shock obliquity

Shock compression x4 in all cases

\[ \gamma (\alpha) \]

Perpendicular shock

Parallel shock

\[ u = c/10 \]

\[ \nu / \omega_g = 0.03 \]

\[ \nu / \omega_g = 0.1 \]

\[ \nu / \omega_g = 0.3 \]

\[ \nu / \omega_g = 1 \]

Radio spectral index
In brackets

\[ \nu = \text{collision frequency} \]

\[ \omega_g = \text{Larmor frequency} \]
Spectral index plotted against shock obliquity

Shock compression x4 in all cases

Radio spectral index
In brackets

ν = collision frequency
ω_g = Larmor frequency
Observations
Cosmic Ray spectrum arriving at earth
(Nagano & Watson 2000)

\[ E^{-2.7} \]

Leakage from galaxy accounts for some of difference (Hillas 2005)
Observed radio spectral index v. mean expansion velocity
(following Glushak 1985)
Spectral steepening suggests quasi-perpendicular shocks

For random field orientation, steepening & flattening nearly cancel out

Steepening at high velocity might be due to

1) expansion into Parker spiral

2) magnetic field amplification
   jxB stretches field perpendicular to shock normal
SNR morphology: spectral steepening/flattening

SN1006 (Chandra)
CR electrons (10-100TeV)

Could be:
1) Quasi-perpendicular shocks accelerate fewer CR to high energy
2) Poor injection at quasi-perpendicular shocks

bi-polar emission (Rothenflug et al 2004)
What we think we know, and what we know we don’t know

What we think we know

• Galactic CR are accelerated to $10^{15}$ eV by diffusive shock acceleration by SNR
• Streaming CR amplify the magnetic field which confines CR near shock
• CR spectra are often steeper than $p^{-4}$ ($E^{-2}$)

What we don’t know

• How CR reach $10^{16}-10^{17}$ eV
• When CR are accelerated to what energy at different stages of SNR evolution
• How CR escape SNR without losing energy adiabatically
• When & where non-linear effects are important
• Why typically the spectrum is flatter than $p^{-4}$ in older SNR
• Why is the spectrum so straight?
• Whether second order Fermi acceleration contributes substantially
• Whether perpendicular shocks are good injectors of low energy CR
• How the above applies extra-galactically - the origin of $10^{20}$eV CR