Introduction to Interstellar Turbulence

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Further reading

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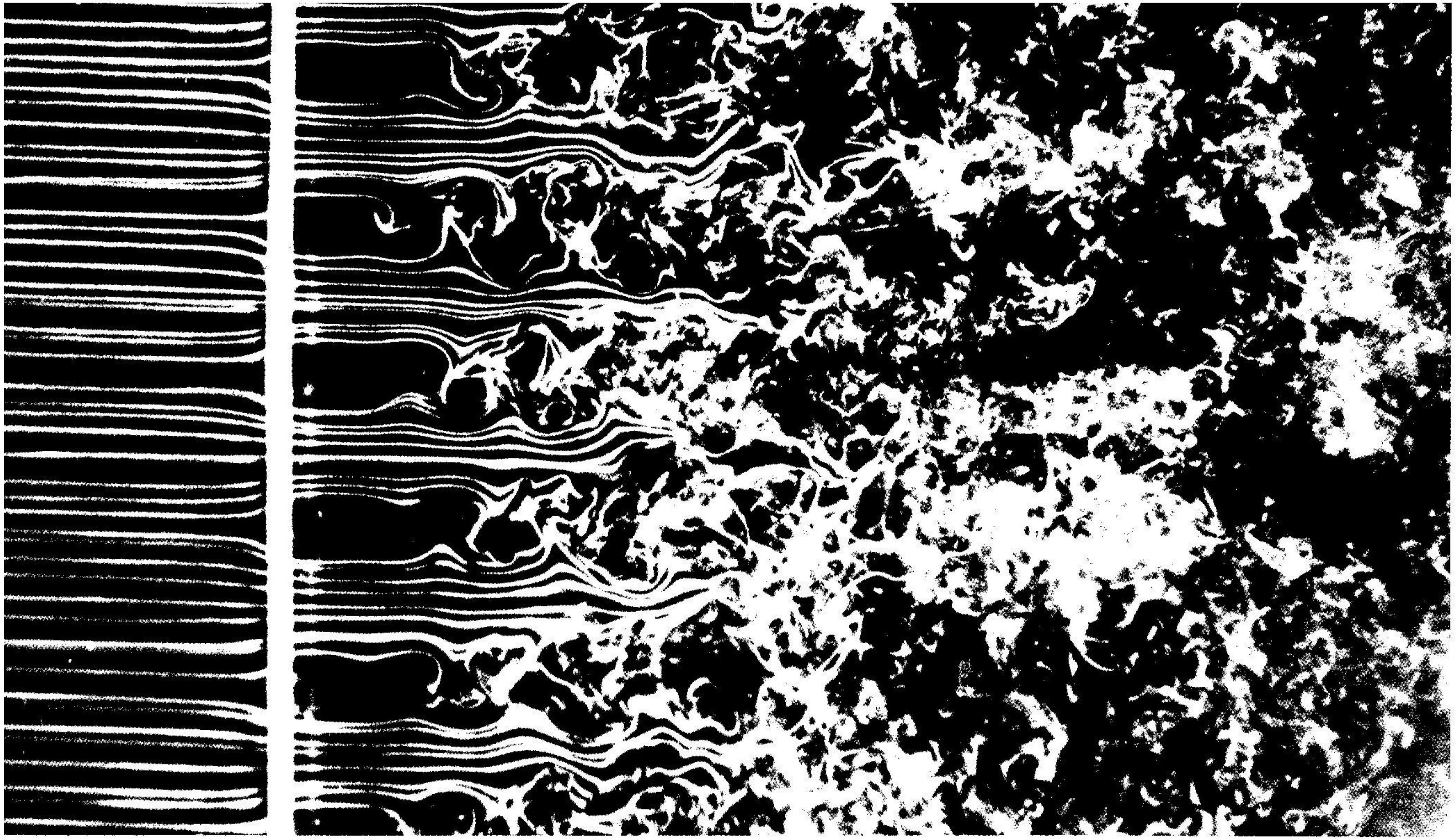
. . ., and references therein

1. Introduction

Flows in nature have the tendency to become disorderly or turbulent

Generation of turbulence by a grid. The Reynolds number is 1500, based on the 1-inch mesh size.

Instability of the shear layers leads to turbulence downstream (Fig. 152 in van Dyke 1982).



Turbulence requires a continuous supply of energy from

- instabilities of a laminar flow (e.g., shear instability, magneto-rotational instability in accretion discs);
- buoyancy, convection, etc.;
- external forces, e.g., supernova explosions in the ISM;
- $\cdots$

Significance of turbulence:

- augments molecular transport and causes mixing within the fluid;
- energy transfer from the large scales of motion: enhanced viscosity, heat transfer, magnetic diffusion;
- generation of coherent structures (flow structures, large-scale magnetic fields via dynamo)

2. An irreducibly short introduction to random functions

Turbulent flows are random

$\Rightarrow$ velocity $\vec{\mathbf{v}}$, pressure p, magnetic field $\vec{\mathbf{B}}$, density ρ are random functions of position $\vec{\mathbf{x}}$ and time t

- $A(x)$ is called a random function of the variable x if $A(x)$ is a random variable for any value x.
- The average $\langle A \rangle$ of $A(x)$: $\quad A = \langle A \rangle + a\ , \quad \langle a \rangle = 0\ .$
- The variance σ_A^2 of $A(x)$: $\quad \sigma_A^2 = \langle (A - \langle A \rangle)^2 \rangle \equiv \langle a^2 \rangle\ .$
- σ_A: the standard deviation (or the r.m.s. value) of A.

- The autocorrelation function of $A(x)$, a measure of relation between neighbouring fluctuations:

$$\begin{aligned} C(x_1, x_2) &= \langle a(x_1) a(x_2) \rangle \\ &= \langle (A(x_1) - \langle A \rangle)(A(x_2) - \langle A \rangle) \rangle \ , \end{aligned}$$

where $\langle A \rangle$ can depend on x.

$$C(x, x) = \sigma_A^2 \ , \qquad C(x_1, x_2) \to 0 \quad \text{for } |x_1 - x_2| \to \infty \ .$$

- The structure function of $A(x)$:

$$D(x_1, x_2) = \langle [a(x_1) - a(x_2)]^2 \rangle \ .$$

- The cross-correlation function of $A_1(x)$ and $A_2(x)$:

$$\begin{aligned} B(x_1, x_2) &= \langle a_1(x_1) a_2(x_2) \rangle \\ &= \langle (A_1(x_1) - \langle A_1 \rangle)(A_2(x_2) - \langle A_2 \rangle) \rangle \ , \end{aligned}$$

- Ensemble, volume, time averaging:

  ergodic random functions are those whose statistical properties obtained by averaging a set of its realizations (ensemble averages) are, with unit probability, equal to those obtained by averaging a single realization for a sufficiently long interval of time (time averages) or a sufficiently large region (volume averages).

- A random function $A(x)$ is called stationary if its mean value and variance are independent of x.

  – Stationary random functions are ergodic
    because different realizations have identical statistical properties.

  – Correlation properties of a stationary random function can be obtained from its single realization.

  – For stationary random functions, with $\delta x = x_1 - x_2$:

$$C(x_1, x_2) = C(\delta x) , \qquad B(x_1, x_2) = B(\delta x) ,$$
$$D(x_1, x_2) = D(\delta x) = 2\langle a^2 \rangle - 2\langle a_1 a_2 \rangle = 2[\sigma_A^2 - C(\delta x)]$$

    $D(\delta x)$ can be calculated from observations or numerical results more accurately and with less computations than $C(\delta x)$.

  – The correlation length: $l_0 = \frac{1}{\sigma_A^2} \int_0^\infty C(\delta x)\, d(\delta x)$.

- Power spectrum (or power spectral density): the Fourier transform of the autocorrelation function,

$$P(k) = \int_{-\infty}^{\infty} C(x) e^{-ikx}\, dx \ , \qquad C(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(k) e^{ikx}\, dk \ .$$

In 3D, $P(\vec{\mathbf{k}}) = \int_V C(\vec{\mathbf{x}}) e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}\, d^3\vec{\mathbf{x}}$ is called the 3D spectrum,
the energy spectrum $E(k)$ is obtained by averaging over all directions in the k-space.

In the isotropic case, $P(\vec{\mathbf{k}}) = P(k)$,

$$E(k)\, dk = \frac{1}{4\pi} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi\, P(k)\, k^2\, dk \quad \Rightarrow \quad E(k) = k^2 P(k) \ .$$

Note on correlation vs. statistical dependence

Cross-correlation: $B_{12}(x_1, x_2) = \langle a_1(x_1) a_2(x_2) \rangle$.

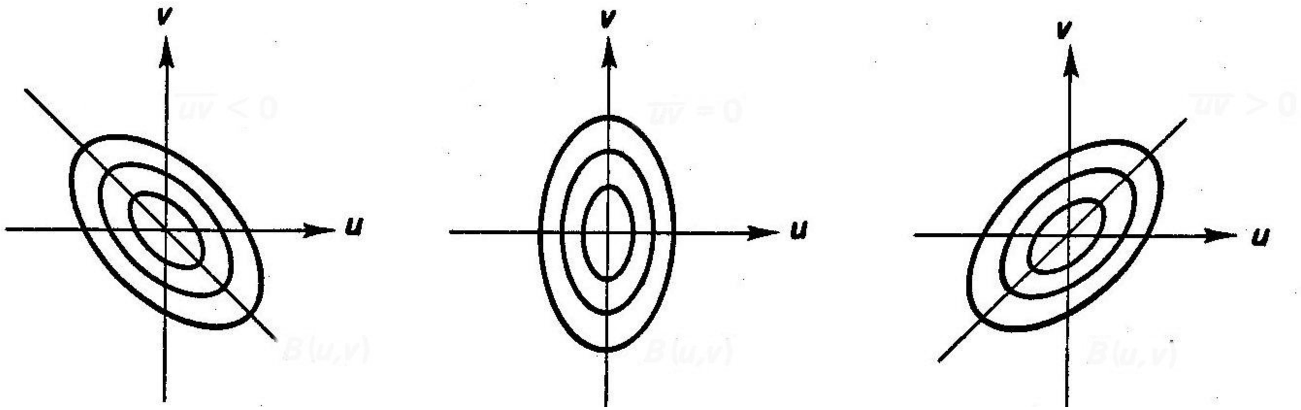
- Positive correlation, $B_{12} > 0$: a_1 large where a_2 is large.
- Anticorrelation, $B_{12} < 0$: a_1 is large where $-a_2$ is large.
- No correlation: $B_{12} = 0$ $\Rightarrow$ $A_1(x)$ and $A_2(x)$ are uncorrelated.

Statistically independent random functions: their joint probability density is equal to the product of their respective probability densities, $p(A_1, A_2) = p_1(A_1) p_2(A_2)$.

- Statistically independent functions are uncorrelated: $B_{12} = \langle a(x_1) \rangle \langle b(x_2) \rangle = 0$.

Contours of joint probability density $p(u, v)$ for random variables u, v that are:

anticorrelated uncorrelated correlated



Contours of joint probability density $p(u, v)$ for random variables u, v that are:

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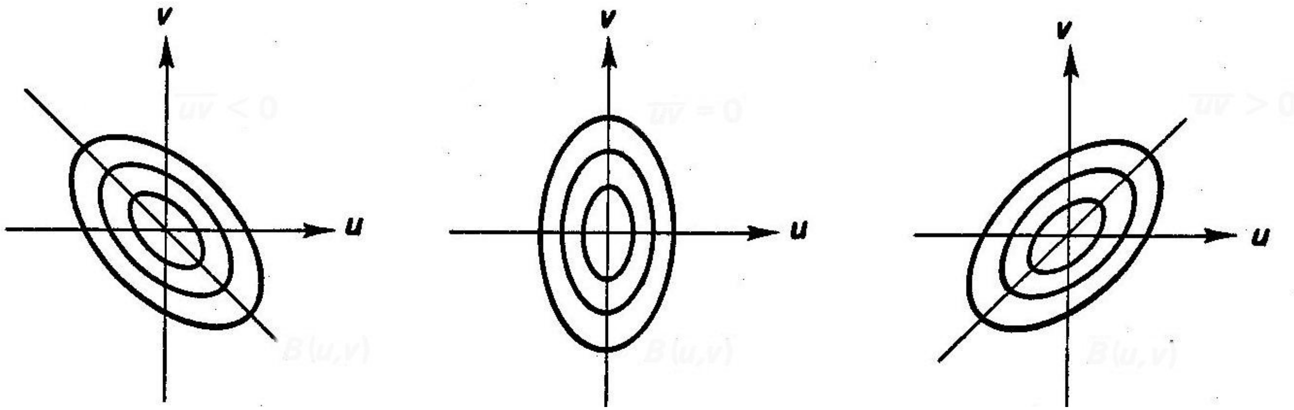
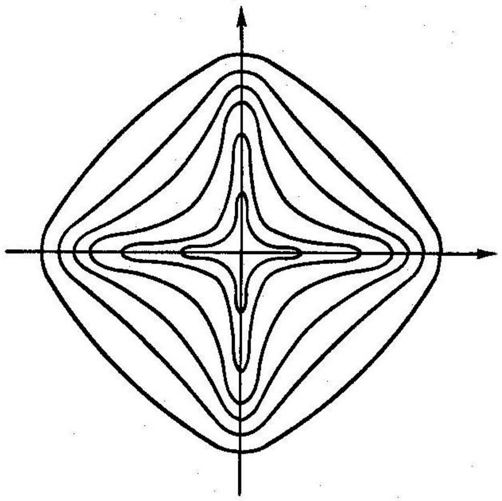


Uncorrelated functions are not necessarily statistically independent.

Contours of $p(u, v)$ for uncorrelated u, v that tend to inhibit each other, and so are statistically dependent on each other: u and v are seldom large (or small) simultaneously.



Question: How can one recognise a factorised function $f(x, y) = g(x)h(y)$ from a plot of its contours, $f(x, y) = \text{const}$?

3. Phenomenology of fluid turbulence

The The Navier–Stokes equation

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \nabla)\vec{\mathbf{v}} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{\mathbf{v}} \ ,$$

known since 1823, probably contains all of turbulence (and much more),
but the nature of turbulence remains one of the most important unsolved problems in physics.

Notation:

$\vec{\mathbf{v}} = \vec{\mathbf{V}} + \vec{\mathbf{u}} =$ velocity,

$\langle \vec{\mathbf{v}} \rangle = \vec{\mathbf{V}} =$ mean velocity;

$\rho =$ density;

$p =$ pressure;

$\nu =$ kinematic viscosity;

$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \vec{\mathbf{b}} =$ magnetic field,

$\langle \vec{\mathbf{B}} \rangle = \vec{\mathbf{B}}_0 =$ mean magnetic field;

$\vec{\mathbf{V}}_{\mathrm{A}} = \frac{\vec{\mathbf{B}}}{\sqrt{4\pi\rho}} =$ the Alfvén velocity.

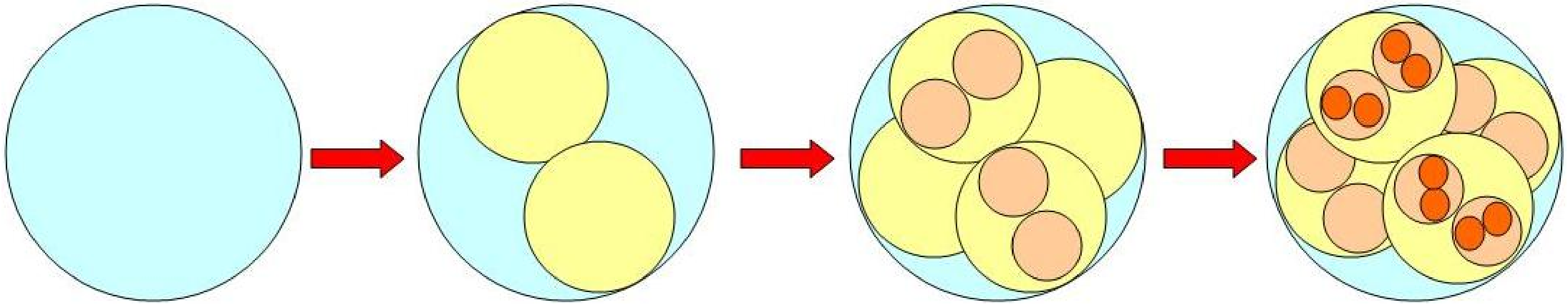
Spectral energy transfer by nonlinear interactions

From a single Fourier mode $v = \sin kx$, the nonlinear term produces another mode:

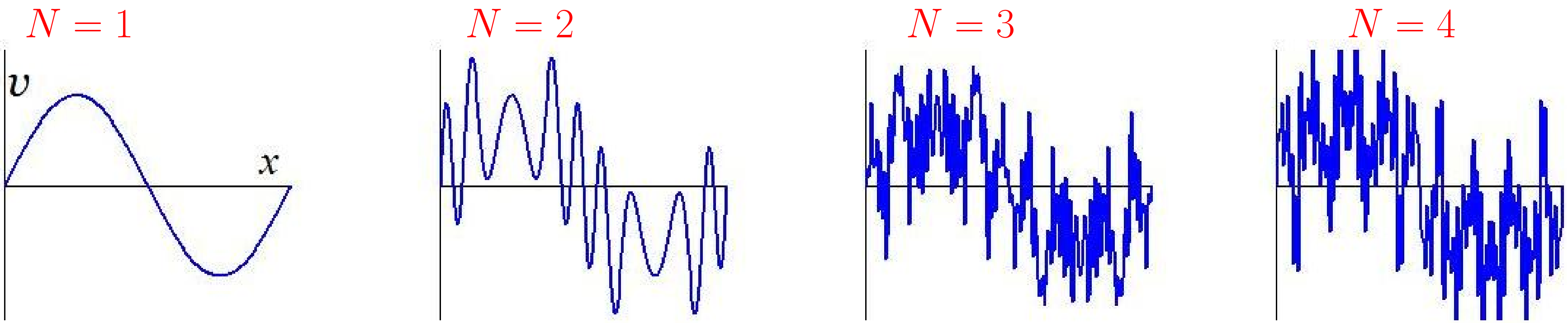
$$(\vec{\mathbf{v}} \cdot \nabla)\vec{\mathbf{v}} = v \frac{\partial}{\partial x} v = k \sin(kx) \cos(kx) \propto \sin(2kx) \ ,$$

so the intertia force drives small-scale motions, i.e., transfers kinetic energy to smaller and smaller scales, from wavenumber k to $2k$, then from $2k$ to $4k$, etc.,

resulting in the energy cascade in the k-space towards small scales:



Flow complexity increases with the range of scales involved $v = \sum_{n=0}^{N-1} k_n^{-1/3} \sin(2\pi\, k_n x) \ , \quad k_n = 2^n$



$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \nabla)\vec{\mathbf{v}} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{\mathbf{v}} \ ,$$

The cascade extends over a broad range of k if viscosity is small:

$$|(\vec{\mathbf{v}} \cdot \nabla)\vec{\mathbf{v}}| \gg |\nu \nabla^2 \vec{\mathbf{v}}| \quad \Rightarrow \quad k v^2 \gg \nu k^2 v \quad \Rightarrow \quad \mathrm{Re} = \frac{l v}{\nu} \gg 1 \ ,$$

where $l = 2\pi/k$ is the wavelength (or scale) of the motion,

i.e., the Reynolds number Re must be large for a large number of scales to be involved in the motion, as observed in turbulent flows.

Free shear layers become turbulent when $\mathrm{Re} \gtrsim (3\text{–}5) \times 10^3$.

In the cool ISM, $\mathrm{Re} \simeq 10^5\text{–}10^7$ (Elmegreen & Scalo, 2004a)

$\Rightarrow$ expect the ISM to be turbulent IF only there are suitable forces to drive the turbulence.

3.1. Kolmogorov's spectrum

Incompressible, homogeneous, isotropic fluid turbulence

Spectral description of the turbulent energy cascade:

- $\mathcal{E} = \frac{1}{2}v_0^2 = \int_0^\infty E(k)\,dk$, specific kinetic energy, $[\mathcal{E}] = \mathrm{cm}^2/\mathrm{s}^2$.

- $E(k)\,dk = \frac{1}{2}v^2(k)\frac{dk}{k} = \frac{1}{2}v^2(k)\,d(\ln k) =$ spectral energy density
  (or kinetic energy spectrum, or specific kinetic energy per unit interval of $\ln k$).

- $v_0 = \sqrt{2\mathcal{E}}$, the r.m.s. velocity.

- $v(k) = \sqrt{2kE(k)}$, velocity at wavenumber k , $[E(k)] = \mathrm{cm}^3/\mathrm{s}^2$.

Kinetic energy is conserved in the turbulent cascade at scaels where viscosity is still negligible

$\Rightarrow$ all the energy arriving to k is transferred to larger k

$\Rightarrow$ energy transfer rate along the spectrum is independent of k:

$$\frac{v^2(k)}{\tau} = \varepsilon \; ,$$

$\varepsilon =$ const, energy transfer rate ,

$\tau =$ time scale of the energy transfer.

$$\tau = \frac{l}{v} = \text{turnover time of an eddy of size } l \quad \Rightarrow \quad \tau = \frac{1}{kv(k)} \; .$$

$$\frac{v^2(k)}{\tau(k)} = \frac{v^2(k)}{1/[kv(k)]} = kv^3(k) = \varepsilon \; ,$$

resulting in Kolmogorov's spectrum

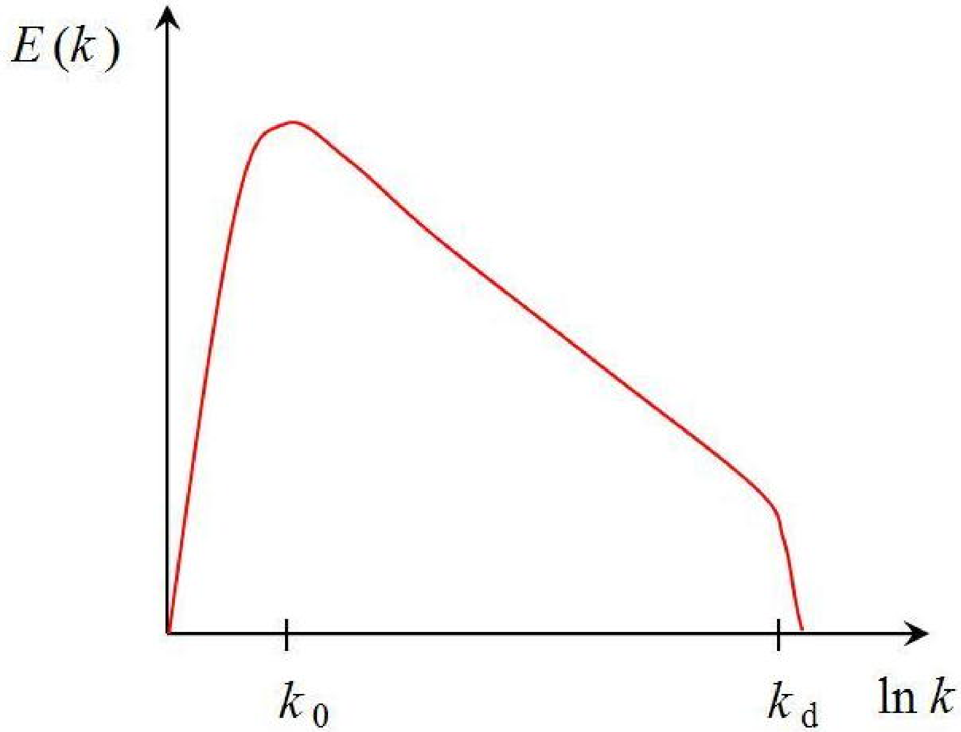
$$\boxed{v(k) = \varepsilon^{1/3} k^{-1/3} \propto l^{1/3} \; , \qquad E(k) = k^{-1} v^2(k) = \varepsilon^{2/3} k^{-5/3} \; ,}$$

up to a dimensionless constant of order unity.

$$v(k) = \varepsilon^{1/3} k^{-1/3} \propto l^{1/3} \,, \quad E(k) = k^{-1} v^2(k) = \varepsilon^{2/3} k^{-5/3} \,, \quad \tau(k) = \frac{1}{k v(k)} \,.$$

Energy transfer rate:

$$\varepsilon = \frac{v^2(k)}{\tau(k)} = \frac{v^2(k_0)}{\tau(k_0)} = k_0 \, v_0^3$$



$k_0 =$ energy injection scale
($=$ integral scale $\approx$ correlation length),

$k_\mathrm{d} =$ dissipation scale,

$k_0 < k < k_\mathrm{d}$, the inertial range
(where the flow is controlled by inertia forces, and where kinetic energy does not dissipate).

Kinetic energy is injected at $k = k_0$ and cascades to larger k, to be dissipated (converted into heat) at $k = k_\mathrm{d}$.

The turbulent cascade terminates at $k = k_\mathrm{d}$ such that

$$|(\vec{\mathbf{v}} \cdot \nabla)\vec{\mathbf{v}}| \simeq |\nu \nabla^2 \vec{\mathbf{v}}| \quad \Rightarrow \quad k_\mathrm{d} v(k_\mathrm{d})^2 \simeq \nu k_\mathrm{d}^2 v(k_\mathrm{d})$$

$$\Rightarrow \quad \frac{v(k_\mathrm{d})}{\nu k_\mathrm{d}} = \mathrm{Re}|_{k=k_\mathrm{d}} \simeq 1 \ .$$

$$\frac{v_0 \, (k_\mathrm{d}/k_0)^{-1/3}}{\nu k_\mathrm{d}} = 1 \quad \Rightarrow \quad \boxed{k_\mathrm{d} = k_0 \mathrm{Re}^{3/4}} \ .$$

The inertial range $k_0 < k < k_\mathrm{d}$ becomes broader with Re.

However small is ν, motions of the integral scale eventually decay at the time scale

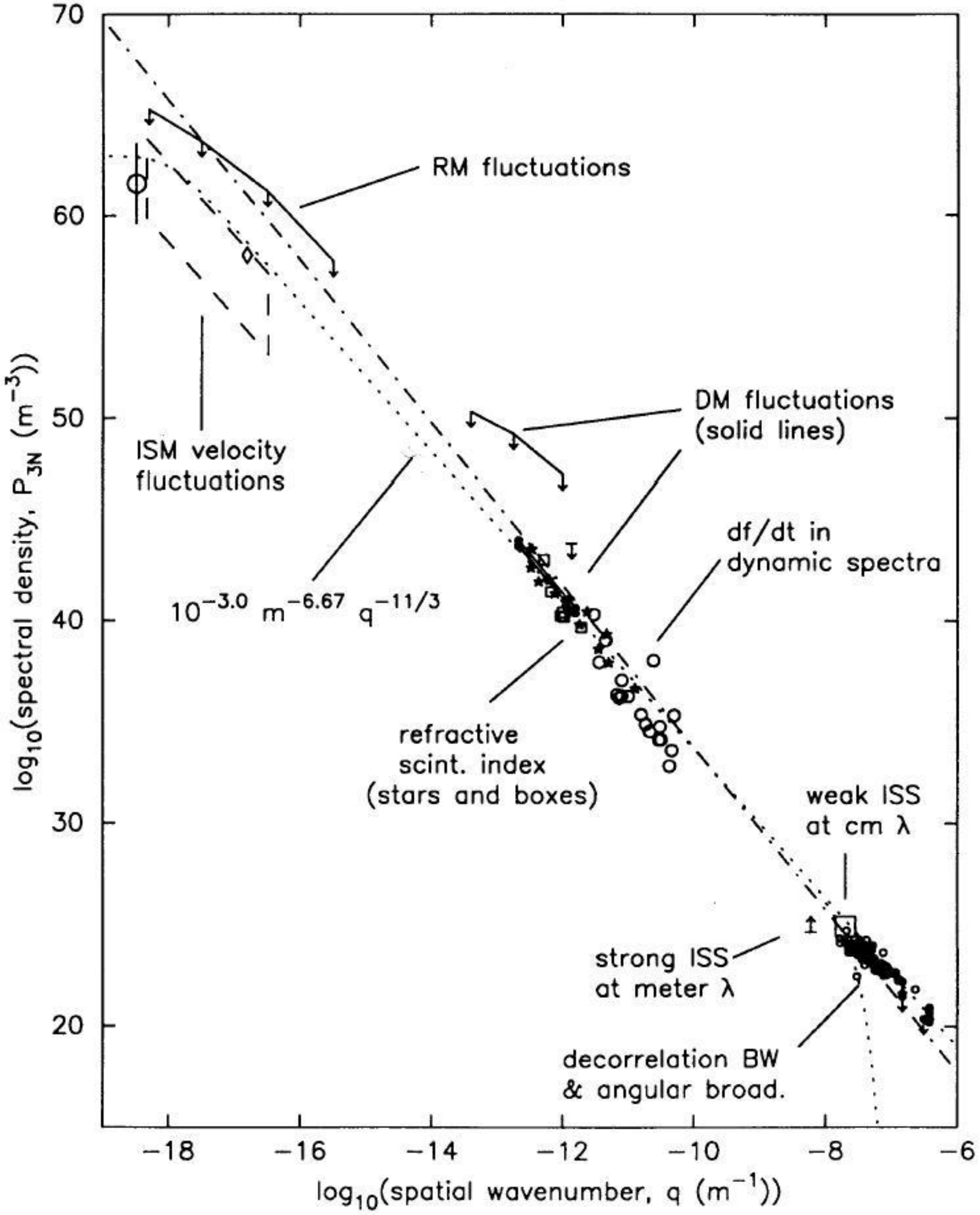
$$\tau_0 \simeq \frac{l_0}{v_0} \ ,$$

the turnover time of the largest eddy:
tubulent motions decay at a time scale independent of viscosity.

Hence, turbulence requires continuous supply of energy even if dissipation is weak.

4. Interstellar turbulence

Big Power Law in the sky: electron density power spectrum in the ISM (Armstrong et al. *ApJ* 433, 209, 1995)



$E(k) \propto k^n,\ n \simeq -5/3$
for $10^{10}\,\mathrm{cm} \lesssim l \lesssim 10^{20}\,\mathrm{cm}$,
or $10^{-3}\,\mathrm{AU} \lesssim l \lesssim 100\,\mathrm{pc}$.

Turbulence observed in/speculated for the ISM:

$l_0 \simeq 100\,\mathrm{pc}$,
$v_0 \simeq c_{\mathrm{sound}} \simeq 10\,\mathrm{km\,s^{-1}}$,
$\tau_0 \simeq l_0/v_0 \simeq 10^7\,\mathrm{yr}$;
$c_{\mathrm{sound}} =$ speed of sound ($T = 10^4\,\mathrm{K}$).

4.1. Energy sources

(Mac Low & Klessen 2004)

- Turbulent kinetic energy density:

$$E = \tfrac{1}{2}\rho v_0^2 \simeq 10^{-12}\frac{\rm erg}{\rm cm^3}\left(\frac{n}{1\,{\rm cm^{-3}}}\right)\left(\frac{v_0}{1\,{\rm km\,s^{-1}}}\right)^2 \quad .$$

- Magnetic energy density: $M = \dfrac{b_0^2}{8\pi} \simeq E$.

- Energy dissipation rate per unit mass:

$$\varepsilon \simeq \frac{v_0^3}{l_0} \simeq 3\times 10^{-3}\,{\rm erg\,g^{-1}\,s^{-1}} \ ,$$

and per unit volume:

$$\varepsilon_V \simeq \rho\frac{v_0^3}{l_0} \simeq 5\times 10^{-27}\,{\rm erg\,cm^{-3}\,s^{-1}} \ ,$$

- Energy dissipation time $=$ largest eddy turnover time:

$$\frac{E}{\varepsilon_V} = \tau_0 = \frac{l_0}{v_0} = 10^7\,{\rm yr}\left(\frac{l_0}{100\,{\rm pc}}\right)\left(\frac{v_0}{10\,{\rm km\,s^{-1}}}\right)^{-1} \quad .$$

Energy dissipation rate per unit volume: $\varepsilon_V \simeq \rho \dfrac{v_0^3}{l_0} \simeq 5 \times 10^{-27}\,\mathrm{erg\,cm^{-3}\,s^{-1}}$.

- Energy sources of the interstellar turbulence

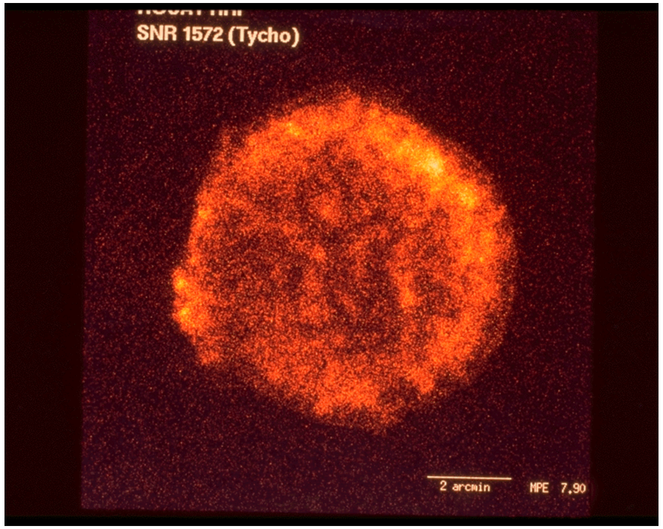
Driving mechanism	ε_V, $\mathrm{erg\,cm^{-3}\,s^{-1}}$
Supernova explosions	3×10^{-26}
Stellar winds	3×10^{-27}
Protostellar outflows	2×10^{-28}
Stellar ionizing radiation	5×10^{-29}
Galactic spiral shocks	4×10^{-29}
Magneto-rotational instability	3×10^{-29}
H II regions	3×10^{-30}

Turbulence driven by supernovae

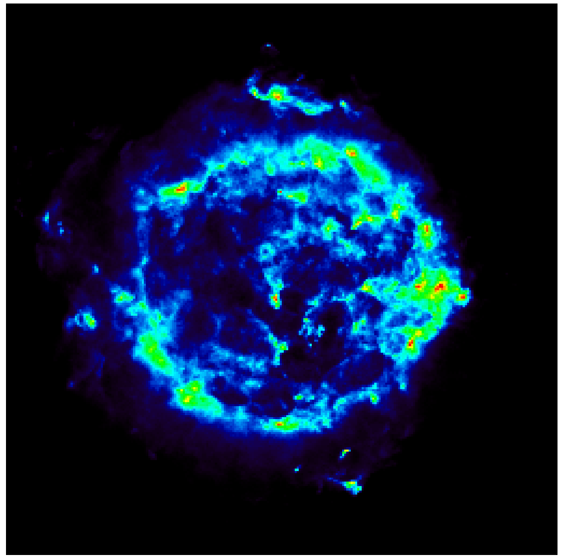
Supernova remnants: expanding bubbles of hot gas, magnetic fields & relativistic particles

Crab nebula: optical image

Tycho supernova: X-rays



Cas A: radio image ($\lambda 6\,\mathrm{cm}$)



Wright et al., Astrophys. J. **518**, 284, 1999

SN explosions:

- energy release $E_{\rm SN} = 10^{51}\,{\rm erg}$ per SN event,
- one type II supernova per 50 years in the Galaxy, frequency $\nu_{\rm SN} = 0.02\,{\rm yr}^{-1}$,
- occur at (quasi) random times and positions.

Supernova blast wave expands at $10^4\,{\rm km\,s^{-1}}$ (Mach 10^3 for the first 300 yr),
then pressure equilibrium after $10^6\,{\rm yr}$,
and a hot gas bubble of $\simeq 100\,{\rm pc}$ in size

Supernova remnants: expanding bubbles of hot gas
that drive motions in the ambient gas when their expansion speed reduces to the speed of sound

Total energy supply rate: $\varepsilon_{\rm SN} = \dfrac{E_{\rm SN}\,\nu_{\rm SN}}{\mathcal{V}} \simeq 2\times 10^{-25}\,{\rm erg\,cm^{-3}\,s^{-1}}$,

$\mathcal{V} = 2\pi R_* h_* =$ volume of the star forming Galactic disc, $R_* = 16\,{\rm kpc}$, $h_* = 100\,{\rm pc}$.

Energy supply required to maintain the turbulence: $\varepsilon_V \simeq \rho\dfrac{v_0^3}{l_0} \simeq 5\times 10^{-27}\,{\rm erg\,cm^{-3}\,s^{-1}}$,

$\simeq 3$% of the energy supplied by the SNe is sufficient to drive the interstellar turbulence.

Turbulent scale = SNR radius in pressure balance with ambient medium

Pressure balance: the (end of the) momentum-conserving (snowplough) phase.

The beginning of the snowplough phase: (Dyson & Williams, *The Physics of the Interstellar Medium,* IOP, 1997, §7.3.4)

- SNR age, $t_0 \simeq 4\times 10^4\,\mathrm{yr}$,
- SNR radius, $r_0 \simeq 25\,\mathrm{pc}$,
- expansion velocity, $\dot r_0 = 250\,\mathrm{km\,s^{-1}}$,
- dense, cool shell of interstellar gas swept up by the SNR.

SNR expansion:

$$r = r_0\left[1+4\,\frac{\dot r_0}{r_0}(t-t_0)\right]^{1/4}, \qquad \dot r = \dot r_0\left[1+4\,\frac{\dot r_0}{r_0}(t-t_0)\right]^{-3/4}.$$

$\dot r = c_\mathrm{sound} = 10\,\mathrm{km\,s^{-1}} \quad \Rightarrow \quad r = l_0$

$$\dot r = c_\mathrm{sound} \quad \Rightarrow \quad 1+4\,\frac{\dot r_0}{r_0}(t-t_0) = \left(\frac{c_\mathrm{sound}}{\dot r_0}\right)^{-4/3}$$

$$\Rightarrow \quad r = r_0\left(\frac{c_\mathrm{sound}}{\dot r_0}\right)^{-1/3} \simeq 70\,\mathrm{pc}\ .$$

Conclusion: the integral scale of the interstellar turbulence is $l_0 = 50$–$100\,\mathrm{pc}$.

Efficiency of SN energy conversion (Dyson & Williams, *The Physics of the Interstellar Medium,* IOP, 1997, §7.3.6)

Kinetic energy of the dense SNR shell:

$$\begin{aligned} E_{\rm shell} &= M_{\rm shell}\,\dot{r}^2 \\ &= \frac{4\pi}{3}\rho_0 r^3 \dot{r}^2 \ , \end{aligned}$$

$M_{\rm shell} =$ mass of the interstellar gas (density ρ_0) swept up by the SNR.

$$t \gg t_0 \quad \Rightarrow \quad r \simeq \left(4 r_0^3 \dot{r}_0 t\right)^{1/4} \ , \qquad \dot{r} \simeq \left(\tfrac{1}{4} r_0^3 \dot{r}_0 t^{-3}\right)^{1/4}$$

Efficiency of SN energy conversion in to kinetic energy:

$$\begin{aligned} \frac{E_{\rm shell}}{E_{\rm SN}} &\simeq \frac{\pi}{3\sqrt{2}E_{\rm SN}}\rho_0 r_0^{15/4}\,\dot{r}_0^{5/4} t^{-3/4} \\ &\simeq 1.2\left(\frac{t_0}{t}\right)^{3/4} \\ &\simeq 8\% \qquad \text{for } t/t_0 = 40 \ . \end{aligned}$$

Conclusions:

- SNe are the most important single source of interstellar turbulence;
- the correlation scale of the turbulence is $l_0 = 50$–$100\,\mathrm{pc}$;
- the turbulent speed is comparable to the speed of sound in the ISM, $v_0 \simeq 10\,\mathrm{km\,s^{-1}}$, or can even exceed it.

4.2. Observational signatures

- Spectral line broadening via Doppler shifts:

$$\Delta\nu_{\rm D} = \nu_0 \left(\underbrace{\frac{2k_{\rm B}T}{m_a c^2}}_{\rm thermal} + \underbrace{\frac{2v_0^2}{3c^2}}_{\rm turbulent} \right)^{1/2} ,$$

$\nu_0 =$ central line frequency,
$k_{\rm B} =$ Boltzmann's constant,
$m_a =$ the emitting atom's mass,
$c =$ speed of light.

Velocity dispersions of interstellar gas scale with the region size l (Larson, *MNRAS,* **186**, 479, 1979; **194**, 809, 1981)

$$\delta v \ ({\rm km\,s^{-1}}) \simeq 1.1 \left(\frac{l}{1\,{\rm pc}}\right)^{\beta} , \qquad \beta = 0.4 \pm 0.1$$

consistently with Kolmogorov's law $v(l) \propto l^{1/3}$.

However, the interpretation of the scaling is controversial (Mac Low & Klessen 2004)

More recently: various statistical studies of velocity and density fluctuations, especially in molecular clouds (Elmegreen & Scalo 2004a)

- Radio wave scattering at electron density fluctuations

  $\Rightarrow$ scintillation, pulse broadening of pulsar emission

  Density fluctuations in weakly compressible turbulence:

$$\frac{\delta n}{\langle n \rangle} \simeq \frac{\delta v}{v_0} \,,$$

  density fluctuations have the same power spectrum as v.

  Significant effects at small scales, $l \lesssim 10^{15}\,\mathrm{cm}$.

5. Magnetohydrodynamic turbulence

Interstellar medium is magnetized, with energy density of magnetic field comparable to the kinetic energy density of turbulence,

$$\tfrac{1}{2}v_0^2 \simeq \frac{1}{8\pi}b_0^2 \simeq 10^{-12}\,\mathrm{erg\,cm^{-3}} = 1\,\mathrm{eV\,cm^{-3}}\,.$$

Furthermore,

$$R_\mathrm{m} \gg \mathrm{Re} \gg 1\,, \qquad R_\mathrm{m} = \frac{lv}{\eta}\,,$$

R_m = magnetic Reynolds number,

η = magnetic diffusivity, $[\eta] = \mathrm{cm^2\,s^{-1}}$.

Thus, interstellar turbulence is a magnetohydrodynamic turbulence

(also, the solar/stellar wind turbulence, turbulence in radio galaxies and quasars, etc.)

5.1. Isotropic Alfvén wave turbulence

(Iroshnikov, *Sov. Astron.*, **7**, 566, 1964;
Kraichnan, *Phys. Fluids*, **8**, 1385, 1965)

Governing equations: The Navier-Stokes equation (**??**) and the induction equation for magnetic field:

$$\frac{\partial \vec{\mathbf{B}}}{\partial t} = \underbrace{\nabla \times (\vec{\mathbf{V}} \times \vec{\mathbf{B}})}_{\text{advection, stretching, compression}} + \underbrace{\eta \nabla^2 \vec{\mathbf{B}}}_{\text{diffusion, decay}} .$$

A convenient variable, Alfvén speed at a scale $l = 2\pi/k$:

$$v_{\mathrm{A}}(k) = \frac{b(k)}{\sqrt{4\pi\rho}} .$$

Kinetic energy density: $E = \frac{1}{2}\rho v^2$

Magnetic energy density: $M = \frac{1}{2}\rho v_{\mathrm{A}}^2$

Kolmogorov turbulence versus isotropic Alfvén wave turbulence

Fluid turbulence	**Alfvén wave MHD turbulence**
Specific kinetic energy: $\mathcal{E} = \frac{1}{2}v_0^2$	Specific magnetic energy: $\mathcal{M} = \frac{1}{2}{v_{\rm A0}}^2$
Kinematic viscosity ν	Magnetic diffusivity η
Kinetic energy spectrum: $E(k) = \frac{1}{2}k^{-1}v^2(k)$	Magnetic energy spectrum: $M(k) = \frac{1}{2}k^{-1}v_{\rm A}^2(k)$
Turbulent eddy	Alfvén wave riding on magnetic field of the largest scale
Constant spectral energy flux: $\dfrac{v^2(k)}{\tau(k)} = \varepsilon$	Constant spectral energy flux: $\dfrac{v^2(k)}{\tau_m(k)} = \dfrac{v_{\rm A}^2(k)}{\tau_m(k)} = \varepsilon$
Spectral energy transfer rate: $\tau(k) = \dfrac{1}{kv(k)}$	Spectral energy transfer rate: $\tau_m(k) = \tau(k)\dfrac{\tau(k)}{\tau_{\rm A}(k)}$, $\tau_{\rm A}(k) = \dfrac{1}{kV_{\rm A}} =$ interaction time of Alfvén waves. $V_{\rm A} \geq v_0 \quad \Rightarrow \quad \tau_{\rm A}(k) < \tau(k)$, weak interaction
Kolmogorov's spectrum: $E(k) = \varepsilon^{2/3}k^{-5/3}$	Iroshnikov–Kraichnan spectrum: $v(k) = v_{\rm A}(k) = (v_{\rm A0}\varepsilon)^{1/4}k^{-1/4}$, $E(k) = M(k) = (v_{\rm A0}\varepsilon)^{1/2}k^{-3/2}$, equipartition between kinetic and magnetic energies, as in a single Alfvén wave.
Dissipation scale: $\tau(k_{\rm d}) = \nu k_{\rm d}^2 \quad \Rightarrow \quad k_{\rm d} = k_0 {\rm Re}^{3/4}$	Dissipation scale: $\tau_m(k_{\rm dm}) = \eta k_{\rm dm}^2 \quad \Rightarrow \quad k_{\rm dm} = k_0 R_{\rm m}^{2/3}\left(\dfrac{v_0}{v_{\rm A0}}\right)^{1/3}$. $k_{\rm d} \ll k_{\rm dm}$ where $R_{\rm m} \gg {\rm Re}$, e.g., in the ISM. Magnetic spectrum in the ISM extends to smaller scales than the velocity spectrum.

5.2. Anisotropic Alfvén wave turbulence

(Sridhar & Goldreich, *ApJ,* **432**, 612, 1994;
Goldreich & Sridhar, *ApJ,* **438**, 763, 1995)

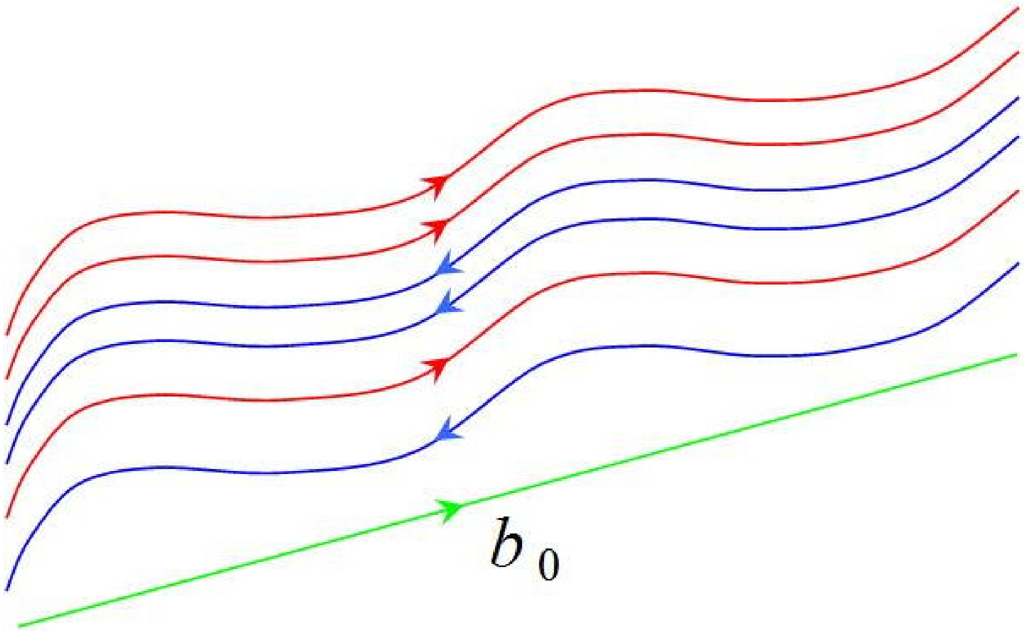
Magnetic field at larger scales introduces anisotropy at smaller scales:

motion along $\vec{\mathrm{b}}_0$ is free, but that across $\vec{\mathrm{b}}_0$ is hindered

⇒ slow variations along the field are allowed, but the wavelength across the field is small:

$$k_\perp \gg k_\parallel \,, \qquad \perp = \text{ perpendicular to } \vec{\mathrm{b}}_0$$

⇒ turbulent “eddies” are elongated along $\vec{\mathrm{b}}_0$.



Balance of energy transfer rates across and along $\vec{\mathrm{b}}_0$:

$$\underbrace{k_\perp v_\mathrm{A}(k_\perp)}_{1/\tau(k_\perp)} \simeq \underbrace{k_\| v_{\mathrm{A}0}}_{\|\ \text{wave frequency}} \tag{1}$$

Spectral energy cascade mainly occurs in the $k_\perp$-plane, with

$$\varepsilon \simeq \frac{v_\mathrm{A}^2(k_\perp)}{\tau(k_\perp)} = k_\perp v_\mathrm{A}^3(k_\perp) \; . \tag{2}$$

Combining (1) and (2), we obtain the aspect ratio of the turbulent cells:

$$l_\| \simeq \frac{v_{\mathrm{A}0}}{\varepsilon^{1/3}} l_\perp^{2/3} \simeq l_0^{1/3} l_\perp^{2/3} \; , \qquad \frac{k_\perp}{k_\|} \simeq (l_0 k_\perp)^{1/3} \; ,$$

with

$$l_0 = \frac{{v_{\mathrm{A}0}}^3}{\varepsilon} \; .$$

⇒ the spectral anisotropy increases with $k_\perp$.

The resulting energy spectrum in the inertial range:

$$E(k_\perp) \;=\; \varepsilon^{2/3} k_\perp^{-5/3} = \left(\frac{{v_{\mathrm{A}0}}^3}{l_0}\right)^{2/3} k_\perp^{-5/3} \; ,$$

$$E(k_\|) \;=\; \varepsilon^{3/2} {v_{\mathrm{A}0}}^{-5/2} k_\|^{-5/2} \; .$$