The magnetic correlation tensor - a short introduction

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Abstract

This document is a written summary of the blackboard talk on the magnetic correlation tensor that was given at the summer school of the DFG research unit 1254 at Ringberg from July 18 to 22, 2011.

1 Introduction

In astrophysics, we often assume magnetic fields to be a turbulent quantity. This essentially means that we have to regard the magnetic field **B** as a random function of which we usually try to compute statistical measures like correlation functions.

Of particular interest is the magnetic autocorrelation function of the magnetic field vector components B_i at some position $\mathbf{x}$ and another position $\mathbf{x}'$

$$\langle B_i(\mathbf{x})B_j(\mathbf{x}')\rangle. \tag{1}$$

The average $\langle \quad \rangle$ should be understood as to be taken over all possible magnetic field configurations according to the statistics of the magnetic fields. The magnetic autocorrelation function can be a very powerful tool in interpreting observations.

For instance, let us assume to have measured a set of rotation measure (RM) values and that we want to calculate their autocorrelation. Then, by the definition of the rotation measure $RM = a_0 \int dz\, n_e(z)B_3(z)$, we can bring this down to a caluculation of the magnetic correlation function (if we assume the electron density n_e to be constant for a moment)

$$\langle RM(\mathbf{x})RM(\mathbf{x}')\rangle \propto \int dz \int dz' \langle B_3(z)B_3(z')\rangle. \tag{2}$$

This particular relationship between $\langle RM(\mathbf{x})RM(\mathbf{x}')\rangle$ and $\langle B_3(z)B_3(z')\rangle$ was e.g. used in the code *REALMAF* (Kuchar & Enßlin 2009) to infer magnetic field power spectra. We also can repeat this kind of exercise using correlations of other radio observables, e.g. to derive the helicity estimator *LITMUS* (Junklewitz & Enßlin 2011).

Thus, having convinced ourselves that the magnetic correlation function is an important quantity, we want to derive its most general form under certain conditions. To concentrate on the fundamentals, we restrict all our statements to the assumption that our random magnetic field is governed by

homogeneous & isotropic turbulence.

2 The magnetic correlation tensor

Let us start defining a correlation function $M_{ij}(\mathbf{x}, \mathbf{x}')$:

$$M_{ij}(\mathbf{x}, \mathbf{x}') = \langle B_i(\mathbf{x}) B_j(\mathbf{x}') \rangle. \tag{3}$$

If we now want to further constrain the analytical form of $M_{ij}(\mathbf{x}, \mathbf{x}')$, we have to somehow apply the assumptions that underly all our derivations. We thus try to formulate the mathematical consequences of a **homogeneous** and **isotropic** random field.

1. **Homogeneity**: If we want our random field to obey homogenous statistics, we actually enforce *translational invariance* on the correlation. That means that the correlation function depends only on the separation $\mathbf{r} = \mathbf{x}' - \mathbf{x}$ of the two positions $\mathbf{x}$ and $\mathbf{x}'$ and not any more on the positions itself (see Figure 1).

2. **Isotropy**: In an analogous way, isotropic statistics will lead to *rotational invariance* of the correlation. For this, we define two arbitrary directions, denoted by two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ (again, see Figure 1) , and describe the magnetic field vectors no longer with respect to fixed coordinate axes but to $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. We therefore take the scalar products

$$\begin{aligned} &a_i B_i(\mathbf{x}) \\ &b_i B_i(\mathbf{x}') \end{aligned} \tag{4}$$

   to project $\mathbf{B}$ on $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

   We now define a correlation scalar $M(\mathbf{r}, \hat{\mathbf{a}}, \hat{\mathbf{b}})$ and see immediately how it is connected to the correlation function

$$\begin{aligned} M(\mathbf{r}, \hat{\mathbf{a}}, \hat{\mathbf{a}}) &= \langle a_i B_i(\mathbf{x})\; b_j B_j(\mathbf{x}') \rangle \\ &= a_i b_i\; \langle B_i(\mathbf{x}) B_j(\mathbf{x}') \rangle \\ &= a_i b_i\; M_{ij}(\mathbf{r}) \end{aligned} \tag{5}$$

   This scalar provides an ideal description for our rotationally invariant system since it is itself invariant under the systems rotation (rotational invariance means that the whole system of vectors depicted in Figure (1) will stay the same under rotation, thus, $\mathbf{r}, \hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ will also stay the same since they define the sketched correlation structure).

We can use our knowledge on the correlation scalar $M(\mathbf{r}, \hat{\mathbf{a}}, \hat{\mathbf{b}})$ to construct a general form of the correlation that now only will depend on the three vectors $\mathbf{r}, \hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. If we just take *one* vector out of the three, the only invariant scalar that can be constructed is its length $\mathbf{r} \bullet \mathbf{r}$ or $\hat{\mathbf{a}} \bullet \hat{\mathbf{a}}$. If we take *two* vectors, we get in addition the scalar product between the two vectors, e.g. $\mathbf{r} \bullet \hat{\mathbf{a}}$. Finally, if we consider all *three* vectors, we also get the volume of the parallelepiped spanned by the three vectors, $\epsilon_{ijk} a_i b_j r_k$.

To sum up, in the most general case, $M(\mathbf{r}, \hat{\mathbf{a}}, \hat{\mathbf{b}})$ will be of the form

$$\begin{aligned} M(\mathbf{r}, \hat{\mathbf{a}}, \hat{\mathbf{b}}) &= \alpha(r)\; (\mathbf{r} \bullet \hat{\mathbf{a}})\; (\mathbf{r} \bullet \hat{\mathbf{b}}) + \beta(\mathbf{r})\; (\hat{\mathbf{a}} \bullet \hat{\mathbf{b}}) + \gamma(\mathbf{r})\; \epsilon_{\mathbf{ijk}} \mathbf{a_i} \mathbf{b_j} \mathbf{r_k} \\ &= \alpha(r)\; r_i a_i\; r_j b_j + \beta(r)\; a_i b_i + \gamma(r)\; \epsilon_{ijk} a_i b_j r_k \end{aligned} \tag{6}$$

where we used the fact that $M(\mathbf{r}, \hat{\mathbf{a}}, \hat{\mathbf{b}})$ is bilinear in $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ and collected all the dependece on $\mathbf{r} = r$ in the three factors α, β and γ.

By comparing to equation (5), we now can formulate a generic form for the magnetic correlation function, from now on called the magnetic correlation tensor to follow standard notation:

$$M_{ij}(\mathbf{r}) = \alpha(r)\, r_i r_j + \beta(r)\, \delta_{ij} + \gamma(r)\, \epsilon_{ijk} a_i b_j r_k \tag{7}$$

At this point we need to constrain the correlation tensor further in order to specify the unknown prefactors. We would do this by demanding the solenoidal condition $\nabla \bullet \mathbf{B} = 0$ to be fullfilled. But this is achieved much easier in Fourier space and furthermore the form of the correlation tensor, which is widely in use, is the one given in Fourier space and therefore in terms of magnetic power spectra. We thus will conclude this presentation by discussion shortly the change to Fourier space.

Actualy, this is done very easily. Since we never specified in which space we are actually working, the above derivation is as well totally valid in Fourier space. Thus, simply by replacing $\mathbf{r}$ with its conjugated Fourier variable $\mathbf{k}$, we already are done.

The solenoidal condition now translates into the mathematically very convenient form

$$k_j M_{ij}(\mathbf{k}) = 0 \tag{8}$$

Using this condition and after some minor work with Fourier Transforms, we finally can write down the standard result for the magnetic correlation tensor in Fourier space:

$$\hat{M}_{ij}(\mathbf{k}) = \hat{M}(k)\, (\delta_{ij} - \frac{k_i k_j}{k^2}) - i\, \hat{H}(k)\, \epsilon_{ijm}\, \frac{k_m}{k} \tag{9}$$

where $\hat{M}(k)$ is the 3D power spectrum of the normal (non-helical) magnetic field. It can be directly related to a 1D energy spectrum ϵ_B which we usually would assume to be some turbulence spectrum. $\hat{H}(k)$ is the helical spectrum, denoting the energy that is stored in the helical fields, which also can be related to a 1D energy spectrum ϵ_H.

References

Junklewitz, H. & Enßlin, T. A. 2011, 530, A88+

Kuchar, P. & Enßlin, T. A. 2009, ArXiv e-prints

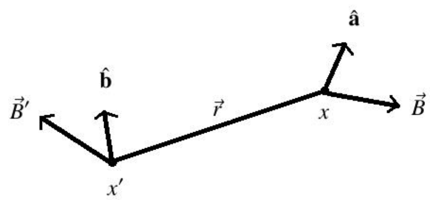

Figure 1: System of vectors that describe the correlation between two positions $\mathbf{x}$ and $\mathbf{x}'$
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