Cowling's Theorem

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> *"And finally, after stewing around and trying to see how I could get through it, I decided that the idea [of the Larmor theory] was wrong, and published the theorem about the impossibility of self-maintained symmetric dynamos."* [*]

1 Brief Introduction

It was around that period of time that Sir Joseph Larmor asked the famous question "How could a rotating body such as the sun become a magnet?" that the notion of the Earth's dipolar magnetic field, as arising from permanently magnetized materials, was shown to be incorrect; the temperature of the earth's core was much higher than the Curie point of the respective metal constituents. Larmor's 1919 theory, regarding the upkeep of sunspot magnetic fields by the currents the fields induced in moving matter was accepted as *the* explanation for magnetic field regeneration in general. [1]

When Cowling first published his theorem in 1934, it caused quite a stir among the physics community. The theorem heralded a paradigm shift from that of the Larmor theory and necessitated a critical revision of the proposed mechanism responsible for the self-maintenance of magnetic fields in the Earth and Sun. It meant that the measured small deviations from axisymmetry (e.g., in the Earth's magnetic field) could no longer be neglected and, in fact, played a crucial role for *large scale* magnetic field regeneration.[2]

Thus, a certain degree of complexity of the magnetic field was required and the prevailing notion of a steady axisymmetric dynamo was abandoned. [3]

[*] excerpt from the AIP Center for History of Physics 1978 oral history interview with Thomas G. Cowling

[1] Sir J. Larmor, "How could a rotating body such as the sun become a magnet?" Brit. Assn. Adv. Sci. Rep. p. 159-160, (1919).

[2] Earth's magnetic field is inclined 11.3° relative to the rotation axis and has both short and long term surface magnetic field fluctuations, the former are known as *geomagnetic secular variation*. In addition, there is a westward drift of the non-dipole field as well as field polarity reversals which occur with a frequency of about every quarter million years.

[3] Karl-Heinz Rädler's article on Mean-Field Dynamos, in particular the section on *Mean-field models of the geodynamo* has a nice short discussion regarding non-axisymmetric magnetic fields of Uranus and Neptune. Available electronically at www.aip.de/People/khraedler/ENCYCL_ 06_ Rae.pdf

Cowling's Theorem is an example of an Anti-dynamo theorem and concerns itself with axisymmetric *magnetic* fields as opposed to axisymmetric *velocity* fields. As a side note and useful rule of thumb, a non-axisymmetric velocity field always creates a non-axisymmetric magnetic field but an axisymmetric velocity field may also produce a non-axisymmetric magnetic field (e.g., Ponomarenko dynamo).

2 Mathematical Sketch

The analysis presented in this section follows that of Professor Chris A. Jones' Les Houches Summer School 2007 Notes on Dynamo Theory: [4]

> ***Cowling's Theorem***
> *An axisymmetric magnetic field vanishing at infinity*
> *cannot be maintained by dynamo action.* [5]

Another words, ***no motion can maintain a perfectly axisymmetric magnetic field***.

We have that $\nabla \cdot \mathbf{B} = 0$ from Maxwell's equations and $\nabla \cdot \mathbf{u} = 0$ for an incompressible flow. This allows the introduction of the magnetic and velocity vector potentials $\mathbf{A}$, ψ where only the magnetic vector potential appears here for convenience

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \nabla \cdot \mathbf{A} = 0 \tag{1}$$

where the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$) has been chosen to ensure that there is no gradient of any scalar present in the definition of $\mathbf{A}$.

Next, the **Induction equation**, assuming constant magnetic diffusivity η, is expressed in terms of the magnetic field and magnetic vector potential as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{2a}$$

$$\frac{\partial \mathbf{A}}{\partial t} = (\mathbf{u} \times \nabla \times \mathbf{A}) + \eta \nabla^2 \mathbf{A} + \nabla \phi, \qquad \nabla^2 \phi = \nabla \cdot (\mathbf{u} \times \mathbf{B}). \tag{2b}$$

[4] Available electronically at http://www.maths.leeds.ac.uk/~cajones/LesHouches.html

[5] T.G. Cowling, The magnetic field of sunspots, Mon. Not. R. Astr. Soc., 94, 39-48, (1934). Cowling's theorem has also been proved for curved space-time. S.R.Maiti, Cowling's theorem in non-flat space-time, Mon. Not. R. Astr. Soc., 194, 827-828, (1981).

In the following treatment, both the magnetic and velocity fields are taken to be axisymmetric ($\frac{\partial}{\partial \phi} \equiv 0$), since a non-axisymmetric flow always produces a non-axisymmetric magnetic field and here only axisymmetric magnetic fields are being treated, enabling the following respective axisymmetric field decompositions in cylindrical coordinates

$$\mathbf{B} = B\hat{\phi} + \underline{\mathbf{B_P}} = B\hat{\phi} + \underline{\nabla \times A\hat{\phi}}, \qquad \mathbf{u} = s\Omega\hat{\phi} + \nabla \times \frac{\psi}{s}\phi \tag{3}$$

where B, u are the azimuthal fields ($\hat{\phi}$ direction) and $\mathbf{B_P}, \mathbf{u_P}$ are the poloidal fields ($\hat{r}, \hat{z}$ direction), Ω is the component of the angular velocity in the $\hat{\phi}$ direction, and $s = r\, sin\, \theta$ is the cylindrical radius.

Inserting the respective field decompositions into the Induction equation (2) yields

$$\frac{\partial B}{\partial t} + \underbrace{s(\mathbf{u_P} \cdot \nabla)(\frac{B}{s})}_{\text{advection}} = \underbrace{\eta\,(\nabla^2 - \frac{1}{s^2})B}_{\text{diffusion}} + \underbrace{s\mathbf{B_P} \cdot \nabla\Omega}_{\text{differential rotation}} \quad , \tag{4a}$$

$$\frac{\partial A}{\partial t} + \frac{1}{s}(\mathbf{u_P} \cdot \nabla)(sA) = \eta(\nabla^2 - \frac{1}{s^2})A. \tag{4b}$$

The advection term spatially translates the field while the diffusion term can not produce any field and only contributes to its gradual decay. The most important term for dynamo action is, therefore, the differential rotation term as it stretches out the poloidal field to generate azimuthal field, given that there exist gradients of angular velocity along the field lines. The differential rotation term is also the source term for the azimuthal field in the above equation.

However, the poloidal field itself does not have a source term and will decay since there are only axisymmetric terms present. A term such as $B\hat{\phi} \cdot \nabla\Omega = B(\frac{\partial\Omega}{\partial\phi})$ would be null by the axisymmetric requirement.

It turns out that eq.(4b) can be shown to decay, implying, via eq.(3), that $\mathbf{B_P}$ decays as well. This leaves eq.(4a) without a source term and as eq.(4a) now has the same form as eq.(4b), it decays in tandem. Thus, dynamo action is precluded by the presence of a perfectly axisymmetric magnetic field.

One may also regard each of the two components of the magnetic field as a piece of string joined to form a circle and the requirement of the field being axisymmetric as a perfect rigidity condition on the string. In order to switch between a poloidal/azimuthal configuration to an azimuthal/poloidal configuration, motions that would result in the string's deformation would be necessary but these are not permissible as the string is perfectly rigid.