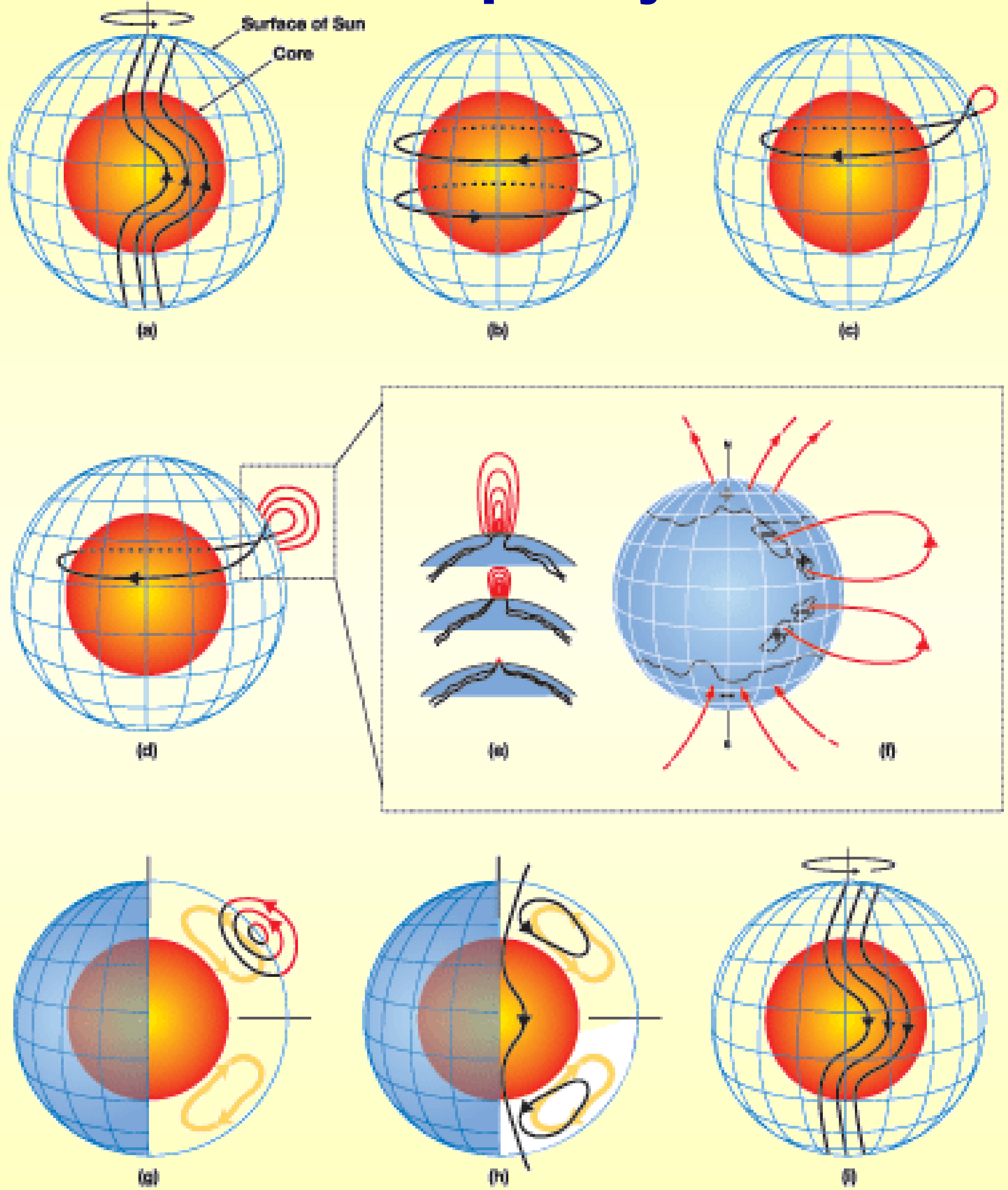
Solar Cycle: ***Theory***

Schematic summary of predictive flux-transport dynamo model

Shearing of poloidal fields by differential rotation to produce new toroidal fields, followed by eruption of sunspots.

Spot-decay and spreading to produce new surface global poloidal fields.

Transport by meridional circulation (conveyor belt) toward the pole, followed by regeneration of new field of opposite sign.



Maxwell's Equations

(Integral Form)

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{inside}}{\varepsilon_0}$$

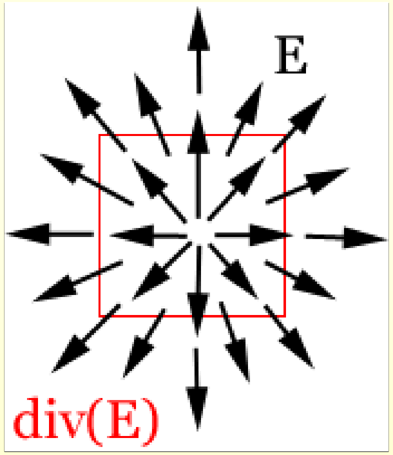
Integral over closed surface

divergence

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

Vector Fields

The divergence of a vector field gives the outward flow from a volume.



$$\mathrm{div}(\mathbf{F}) \equiv \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z)$$

$$\iint_S \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} Q_{enclosed}$$

Gauss' Law

A result from vector calculus, **Gauss' Theorem**, says

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \iiint_V \nabla \cdot \mathbf{F} dV$$

Using a *charge density:*

$$\iiint_V \nabla \cdot \mathbf{E} dV = \frac{1}{\epsilon_0} Q_{enclosed} = \frac{1}{\epsilon_0} \rho dV$$

Taking the limit as V goes to zero

$$\nabla \cdot \mathbf{E} dV = \frac{1}{\epsilon_0} \rho dV$$

The first of Maxwell's Equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Maxwell's Equations

(Integral Form)

divergence

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{inside}}{\varepsilon_0}$$

Integral over closed surface

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

$$\oint \vec{B}\cdot d\vec{A} = 0$$

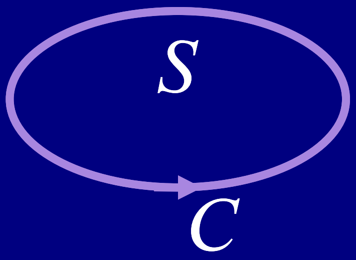
$$\nabla \cdot \mathbf{B} = 0$$

No magnetic monopoles

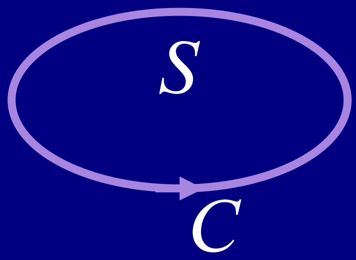
Faraday's Law of Electromagnetic Induction

- "The electromotive force induced around a closed loop ***C*** is equal to the time rate of decrease of the magnetic flux linking the loop."

$$V_{ind} = -\frac{d\Phi}{dt}$$



– The electromotive force



- S is any surface bounded by C

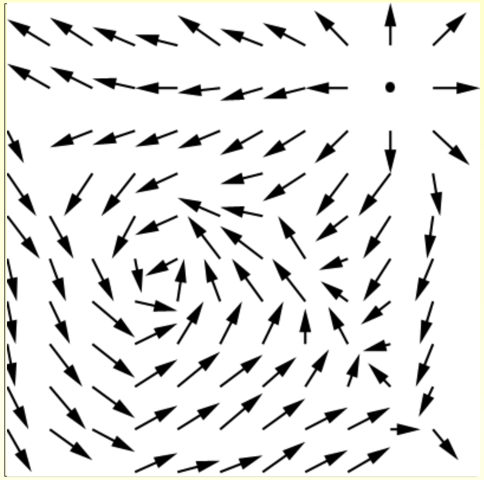
$$V_{ind} = -\frac{d\Phi}{dt}$$

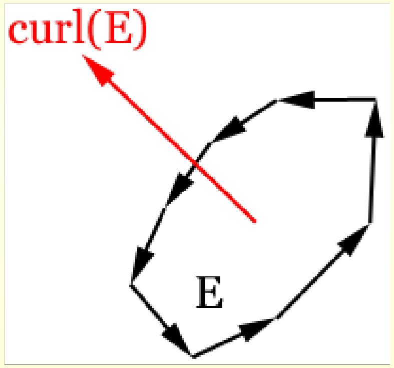
$$\oint_C \underline{E} \cdot d\,\underline{l} = -\frac{d}{dt}\int_S \underline{B} \cdot d\,\underline{s}$$

integral form of Faraday's law

Vector Fields

The curl of a vector field gives the circulation





$$\mathrm{curl}(\mathbf{F}) \equiv \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_x, F_y, F_z)$$

Faraday's Law

As we did for Gauss' Law, shrink *S* to an infinitesimally small surface to get the differential form:

$$\oint_C \mathbf{E} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{E}) \cdot \mathbf{dA}$$
$$= -\frac{d}{dt} \iint_S \mathbf{B} \cdot \mathbf{dA}$$

Faraday's Law of Induction:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Note:

- Electrostatics:

$$\nabla \times \underline{E} = 0 \;\Rightarrow\; \underline{E} = -\nabla \mathfrak{V}$$

scalar electric potential

- Electrodynamics:

$$E = -\nabla \mathfrak{V} - \frac{\partial \underline{A}}{\partial t}$$

vector magnetic potential

scalar electric potential

Maxwell's Equations

(Integral Form)

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{inside}}{\varepsilon_0}$$

Integral over closed surface

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

Integral over closed line (circuit)

divergence

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Ampere's Law

From vector calculus, **Stokes' Theorem** says

$$\oint_C \mathbf{F} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{dA}$$

Apply this, and make the surface infinitesimally small:

$$\oint_C \mathbf{B} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{B}) \cdot \mathbf{dA}$$
$$= \frac{I_{enclosed}}{\epsilon_0 c^2}$$

Differential form of Ampere's Law: $$\nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0 c^2}$$

Maxwell's Equations

(Integral Form)

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{inside}}{\varepsilon_0}$$

Integral over closed surface

$$\oint \vec{B}\cdot d\vec{A} = 0$$

$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

Integral over closed line (circuit)

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{enclosed} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

No magnetic monopoles and no magnetic "currents".

divergence

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

$$\nabla \cdot \mathbf{B} = 0$$

curl

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

The Lorentz Force

The total force on a charged particle due to electric and magnetic fields is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

E'=F/q

Ohm's law

V=I R ⟶ E'= Jη

$$\eta \mathbf{J} = (1/\sigma)\, \mathbf{J} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

where **J** = current density, **E** = electric field, **B** = magnetic flux density; η = electrical resistivity, σ = electrical conductivity.

$$\nabla \times \mathrm{B} = \mu_0 \mathbf{j}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{j} = \sigma_0(\mathbf{E} + \mathbf{v} \times \mathrm{B})$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathrm{B}) + \frac{1}{\mu_0 \sigma_0} \nabla^2 \mathbf{B}$$

Appendix:

Take the curl of both sides:

$$(1/\sigma)\ \text{curl}\ \mathbf{J} = \text{curl}\ \mathbf{E} + \text{curl}\ (\mathbf{v} \times \mathbf{B}) \qquad (2)$$

L.H.S. , by Maxwell's equations

$$\mu_0 \text{curl}\ \mathbf{J} = \text{curl curl}\ \mathbf{B} = \text{grad div}\ \mathbf{B} - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B}$$

R.H.S. is , by Maxwell's equations, $\text{curl}\ \mathbf{E} = -\partial \mathbf{B}/\partial t$

$$(1/\sigma)\ \text{curl}\ \mathbf{J} = -\partial \mathbf{B}/\partial t + \text{curl}\ (\mathbf{v} \times \mathbf{B})$$

So $$\partial \mathbf{B}/\partial t = \nabla^2 \mathbf{B} / (\sigma \mu_0) + \text{curl}\ (\mathbf{v} \times \mathbf{B}) \qquad \mathbf{(3)}$$

Induction equation (Summary)

Faraday's law in combination with the simple phenomenological ***Ohm's law***, relating the electric field in the plasma frame with its current:

$$\mathbf{j} = \sigma_0(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Using ***Ampere's law*** for slow time variations, without the displacement current and the fact that the field is free of divergence (**$\nabla \cdot \mathbf{B} = 0$**),

yields the induction equation (with conductivity σ_0):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma_0} \nabla^2 \mathbf{B}$$

Convection Diffusion

Evolution of a magnetic field in a plasma, with conductivity σ, moving at velocity v

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \frac{1}{\mu_0 \sigma} \nabla^2 B$$

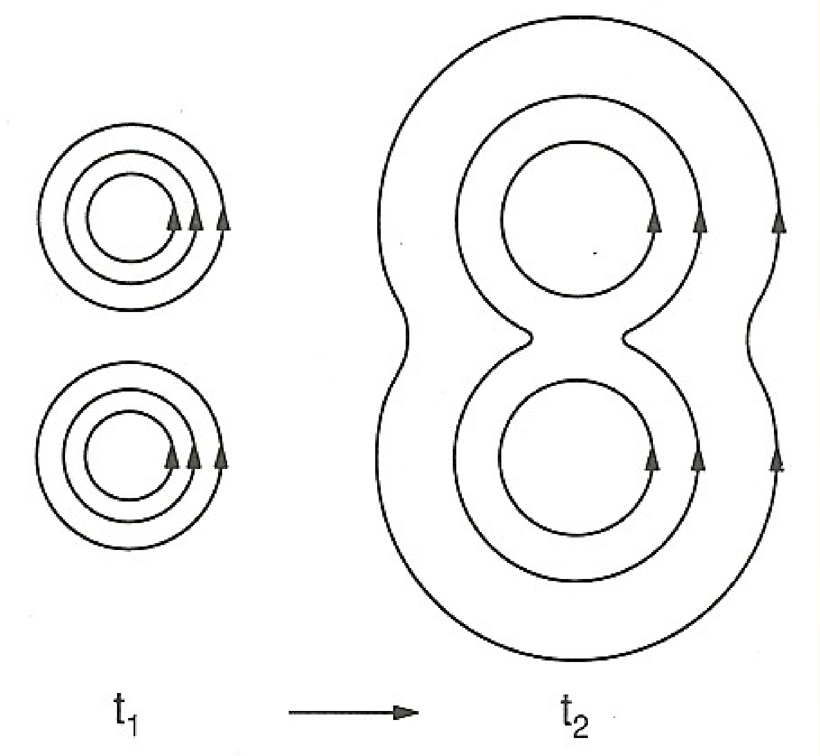
- **The first term of the right hand side describes the behaviour (coupling) of the magnetic field with the plasma**
- **The second term on the right hand side represents diffusion of the magnetic field through the plasma.**

Magnetic diffusion

Assuming the plasma be at rest, the induction equation becomes a pure diffusion equation:

$$\frac{\partial \mathrm{B}}{\partial t} = D_m \nabla^2 \mathrm{B}$$

with the magnetic ***diffusion coefficient*** $D_m = (\mu_0 \sigma_0)^{-1}$.



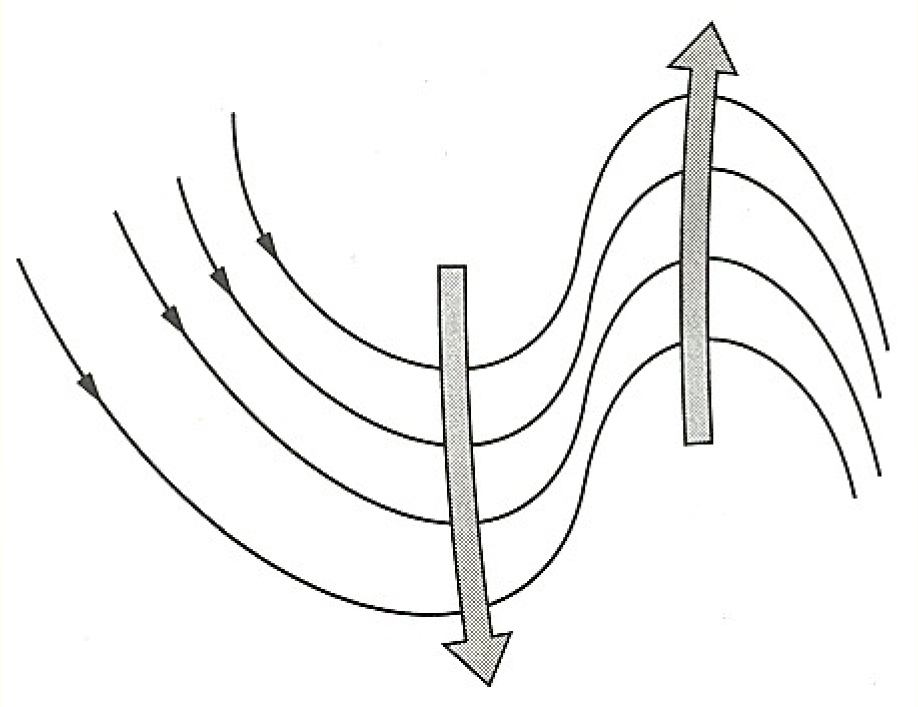
Under the influence of finite resistivity the magnetic field diffuses across the plasma and field inhomogenities are smoothed out at time scale,

$\tau_d = \mu_0 \sigma_0 L_B^2$, with scale length L_B.

Hydromagnetic theorem

For a plasma with infinite conductivity the induction equation becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$



The field lines are constrained to move with the plasma -> ***frozen-in field***. If plasma patches on different sections of a bundle of field lines move oppositely, then the lines will be deformed accordingly.

Schrijver and Zwan

Solar and Stellar Magnetic Activity

Cambridge, Univ. Press 2000

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

$$\mathcal{O}\left(\frac{B}{\hat{t}}\right) \sim \mathcal{O}\left(\frac{vB}{\ell}\right) + \mathcal{O}\left(\frac{\eta B}{\ell^2}\right) \tag{4.7}$$

The magnetic Reynolds number

If the scale length of the plasma is L, the gradient term is (approximately) $\nabla \sim 1/L$

The ratio R_M between the two terms on the right hand side of the induction equation is $R_M \sim \mu_0 \sigma L v$

R_M = Magnetic Reynolds number

- **If $R_M \ll 1$ then the diffusion term dominates**
- **If $R_M \gg 1$ then the coupling term dominates**

Kinematic/Dynamic Dynamo

MHD induction eq.

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}.$$

• If plasma velocity is given, the induction equation is linear and the problem is called kinematic (linear) dynamo.

For a given velocity, solutions of the form

B~exp(a+ib)t are searched.

• When back reaction of the ***B*** on ***U*** is considered, one has to solve momentum equation and hence the problem is nonlinear. It is called dynamic (nonlinear) dynamo.

The first studies trying to describe the real magnetic field <B> were based on rotating bodies in which both magnetic field and velocity field u are axisymmetric

. One of these studies was undertaken by Cowling (1934), who came to the rather disappointing conclusion "that it is impossible that an **axially symmetric field** shall be self-maintained".

A breakthrough came when Parker (1955b) suggested the use of an averaging procedure and to describe the mean magnetic field <B>.

This **mean field** can be steady and axisymmetric since only the real total field

B=<B>+b

(where b are the non-axisymmetric fluctuations in B) is subject to Cowling's theorem.

The new element in this approach is what Parker called **cyclonic motion: the twisting of magnetic field lines by helical convection**.

Then after 1966, Steenbeck, Krause and Rädler published a series of papers (translated in English by Roberts and Stix (1971)), which provided a mathematical bases for Parker's suggestions.

The quintessence is that the velocity field is split in two parts:

U(r,t) = U0+u(r,t), where U0=<u> represents the large-scale motion, and u,

having zero average, is the turbulent velocity.

Kinematic Mean Field Theory

Starting point is the magnetic induction equation of MHD:

$$\frac{\partial B}{\partial t} = \nabla \times (U \times B) + \eta \nabla^2 B,$$

where **B** is the magnetic field, **u** is the fluid velocity and η is the magnetic diffusivity (assumed constant for simplicity).

Assume scale separation between large- and small-scale field and flow:

$$B = B_0 + b, \quad U = U_0 + u,$$

where **B** and **U** vary on some large length scale *L*, and **u** and **b** vary on a much smaller scale *l*.

$$\langle B \rangle = B_0, \ \langle U \rangle = U_0,$$

where averages are taken over some intermediate scale $l \ll a \ll L$.

$$\boldsymbol{\varepsilon} = \langle u \times b \rangle .$$

$$\boldsymbol{\varepsilon} = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \ldots$$

Then mean induction equation becomes:

$$\frac{\partial B_0}{\partial t} = \nabla \times (U_0 \times B_0) + \nabla \times (\alpha B_0) + (\eta + \beta) \nabla^2 B_0 .$$

α: regenerative term: regenerates poloidal fields from toroidal fields by cyclonic turbulence through the twisting action of the Coriolis force on magnetic field loops in the convective cells (alpha effect).

,

β: turbulent diffusivity: Since beta>>eta,
the turbulent diffusion time is much smaller than the ohmic diffusion time.

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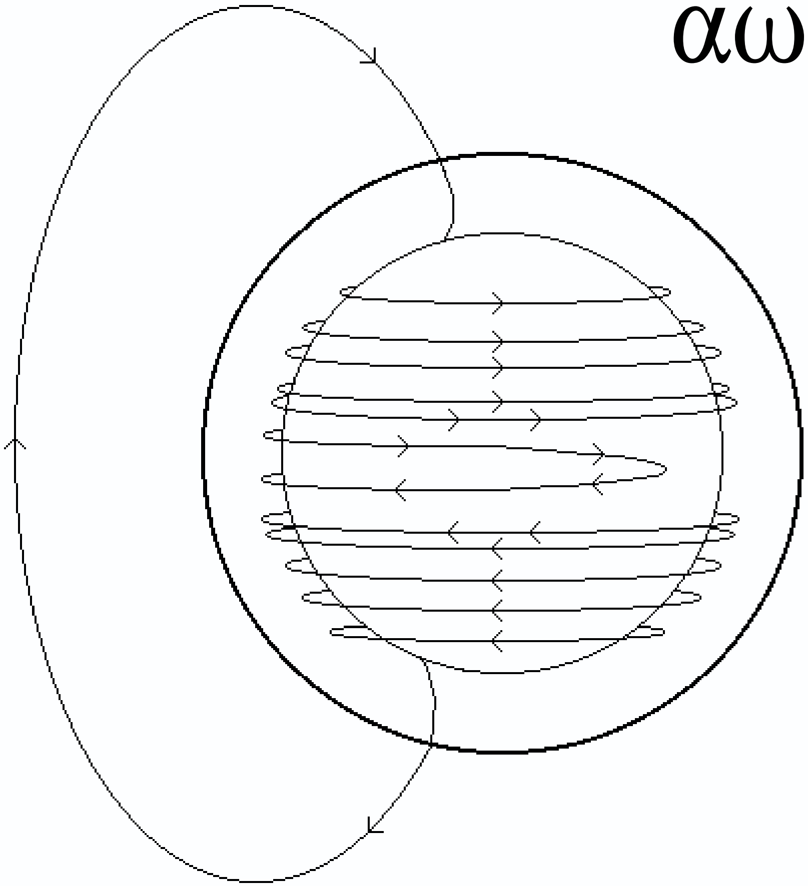
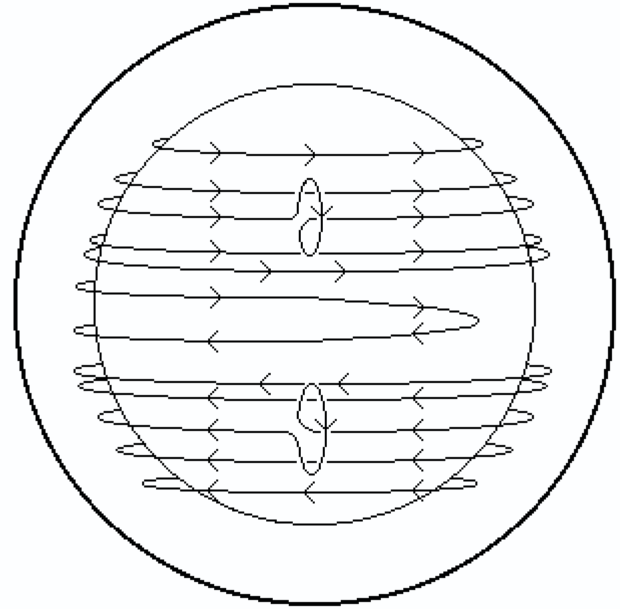
αωdynamo



The ω-effect

Generate toroidal field from poloidal field by stretching the field line by differential rotation.



The α-effect

Generate poloidal field from toroidal field by Colioris force, turbulence , MHD instability etc.

Mathematical Formulation

Maxwell's equations + generalized Ohm's law lead to induction equation :

$$\frac{\partial B}{\partial t} = \nabla \times \left(U \times B - \eta \nabla \times B \right). \qquad \textbf{(1)}$$

Applying mean-field theory to (1), we obtain the dynamo equation as,

$$\frac{\partial Bo}{\partial t} = \nabla \times \left(Uo \times Bo + \alpha Bo - (\eta + \beta) \nabla \times Bo \right),$$

Differential rotation and meridional circulation from helioseismic data

Poloidal field source from active region decay

Turbulent magnetic diffusivity

ASsume axisymmetry, decompose into toroidal and poloidal components:

$$Bo = B_\varphi(r,\theta,t)\hat{e}_\varphi + \nabla \times \left[A(r,\theta,t)\hat{e}_\varphi \right],$$

Toroidal field — Poloidal field

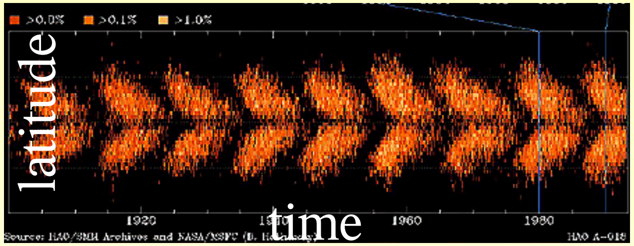
$$Uo = u(r,\theta) + r\sin\theta\,\Omega(r,\theta)\hat{e}_\varphi,$$

Meridional circulation — Differential rotation

The αΩ dynamo

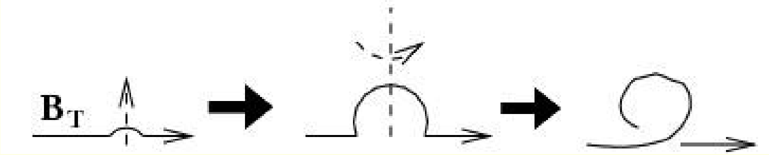
- αΩ-dynamo

$$B = B_T + B_P$$

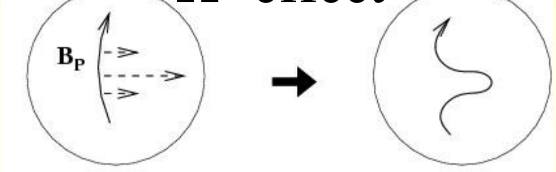


(Courtesy HAO)

α–effect



Ω–effect



How to model…cont’d

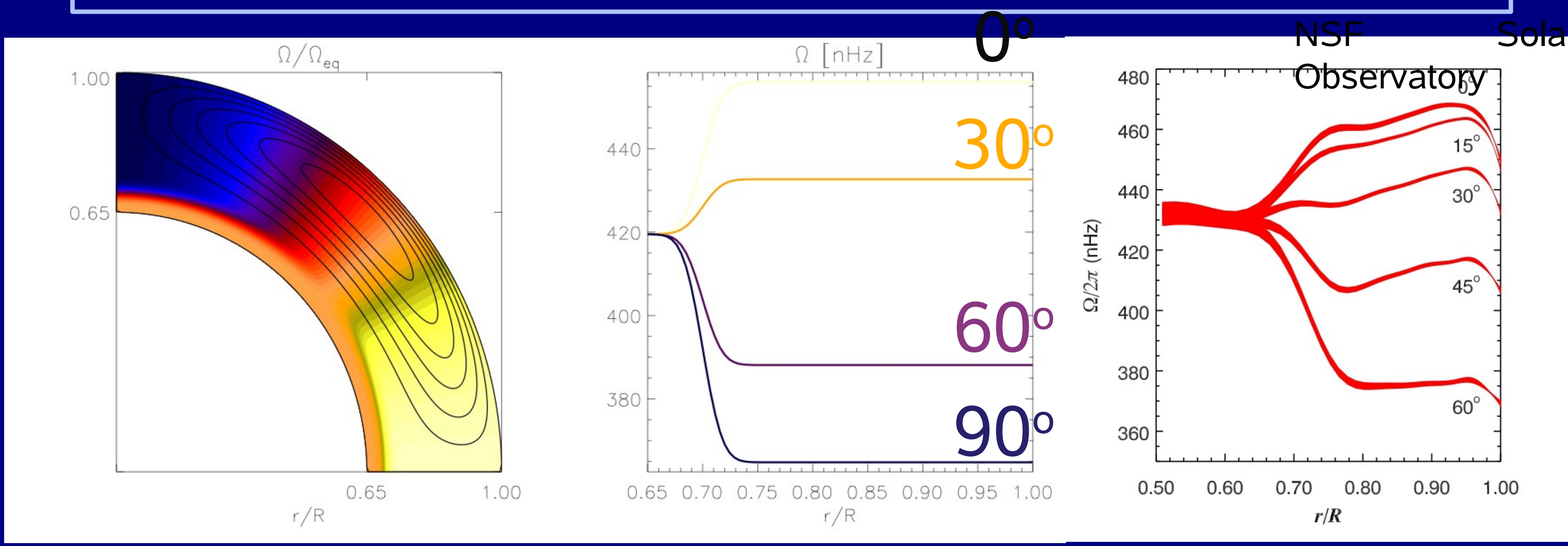
Assume axisymmetry

$$\overline{\mathbf{B}} = \overline{B_\phi}(r,\theta,t)\hat{\mathbf{e}}_\phi + \nabla \times \overline{A_\phi}(r,\theta,t)\hat{\mathbf{e}}_\phi$$

Adopt idealized descriptions of the large-scale velocity field

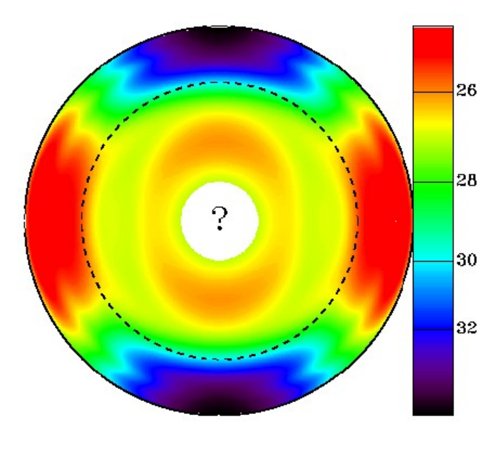
$$\overline{\mathbf{U}} = \overline{\mathbf{U}}_{\rm mer}(r,\theta) + r\sin\theta\,\Omega(r,\theta)\hat{\mathbf{e}}_\phi$$

$$\rho\overline{\mathbf{U}}_{\rm mer} = \nabla \times (\psi\hat{\mathbf{e}}_\phi)$$



Mathematical formulation

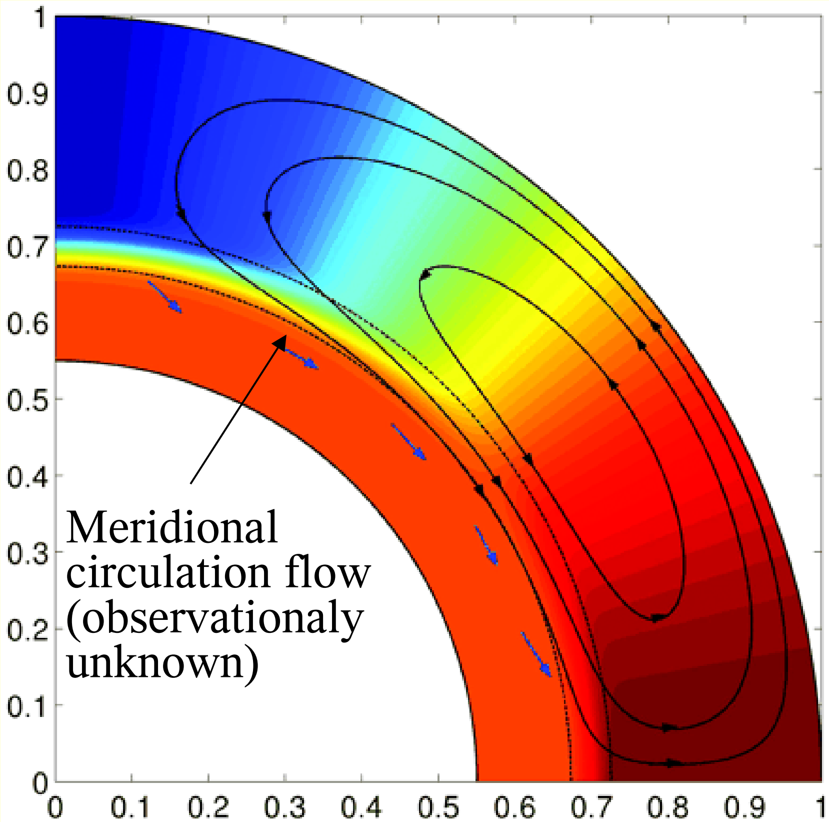
From helioseismology



•Strong velocity shear at the base of the convection zone (tachocline) => ωeffect

•Buoyancy : if $B>10^5$G, the flux is made to erupt to the surface layers.

Model in the simulation

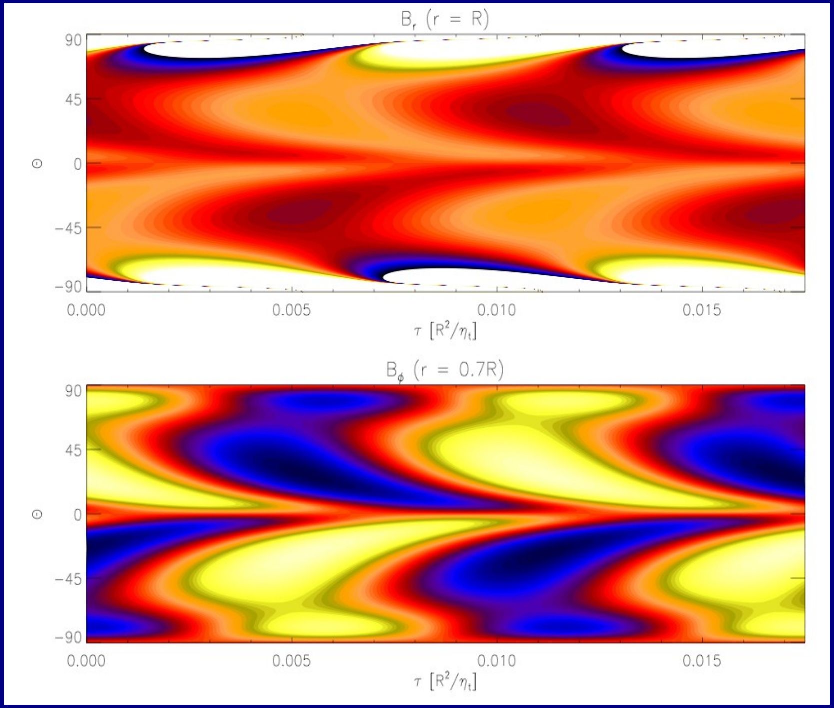


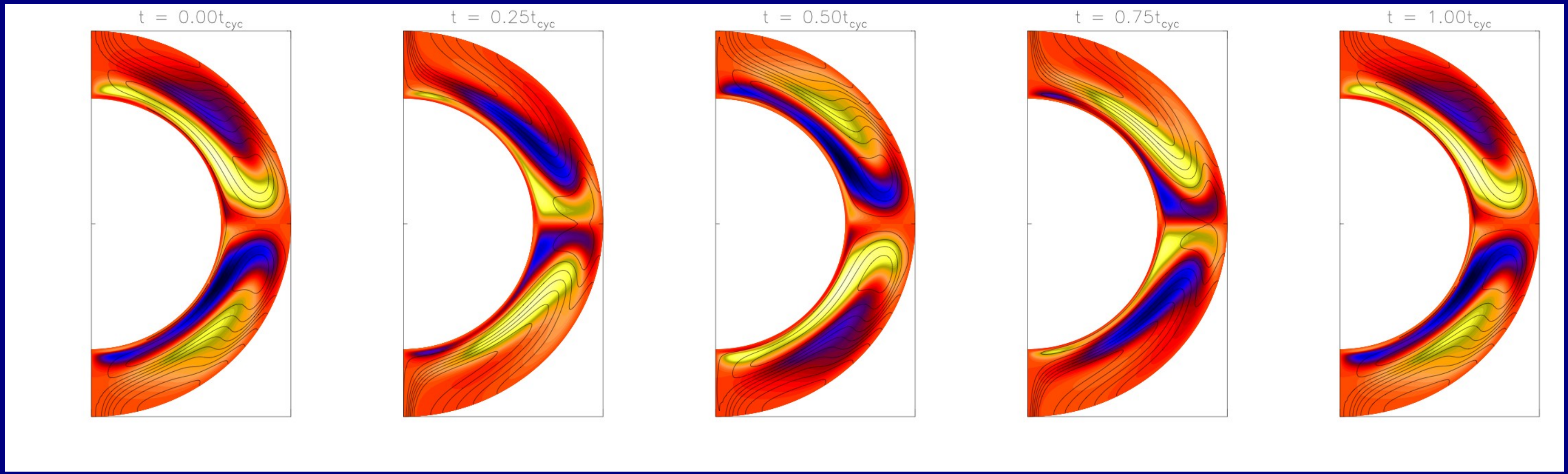
Results

$$\eta_t^{(\mathrm{surf})} = \eta_t^{(\mathrm{bulk})} = 10^7 m^2 s^{-1}$$

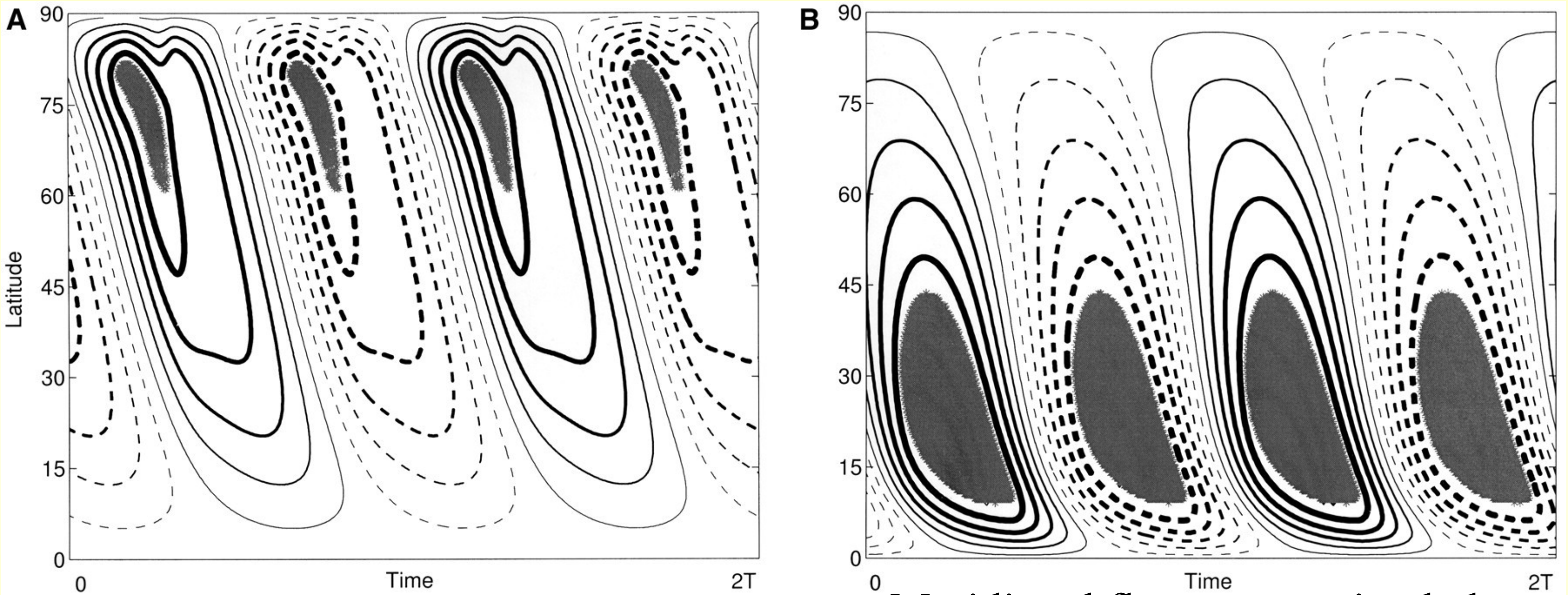
$$\alpha = 0.24\ m\ s^{-1}$$

$$t_{\mathrm{cyc}} = 0.012\tau_{\mathrm{D}} \approx 19yr$$





Eruption latitude vs time plot of sunspots

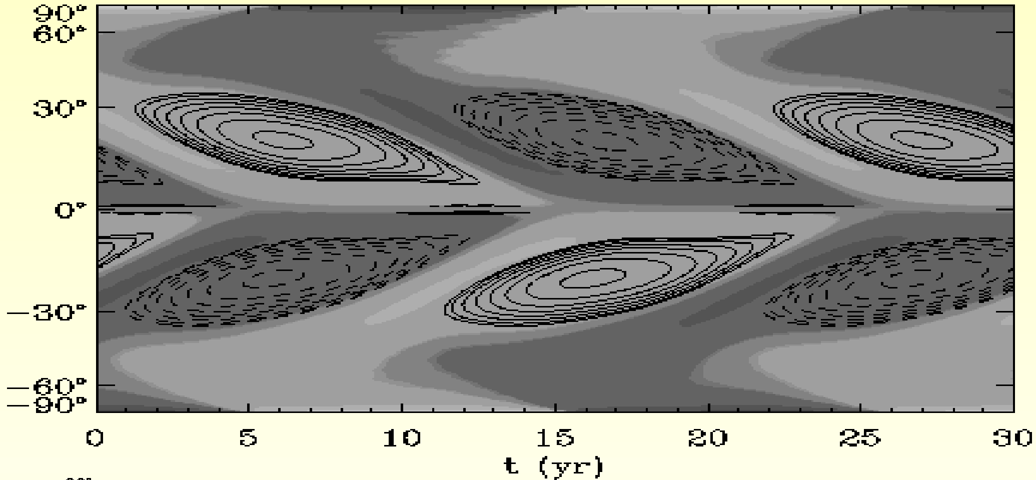


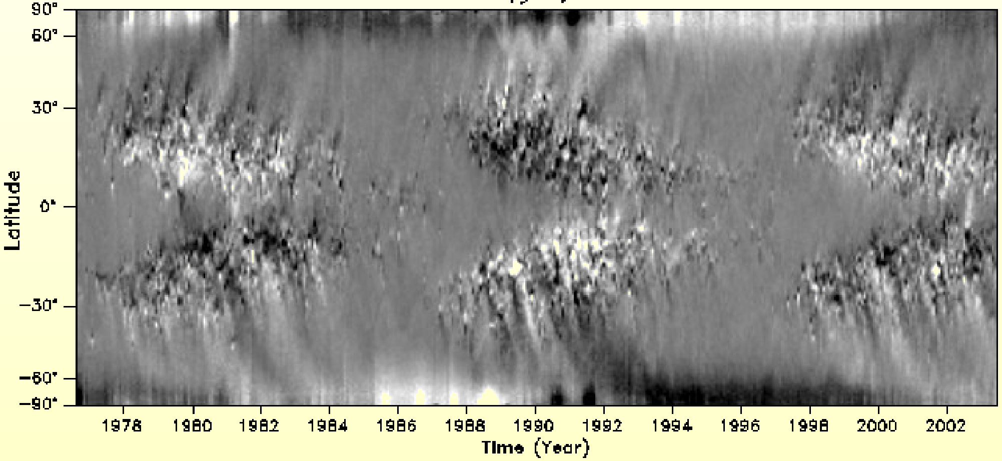
Meridional flow only in the convection zone

Meridional flow penetrating below the tachocline

Sunspots appear in high latitude region (inconsistent with observation) if the Meridional flow does not penetrate into the stable (radiateve) zone.

Validity Test of Calibration





Observed NSO map of longitude-averaged photospheric fields

(Dikpati, de Toma, Gilman, Arge & White, 2004, ApJ, 601, 1136)

Mean Field Theory – Applications

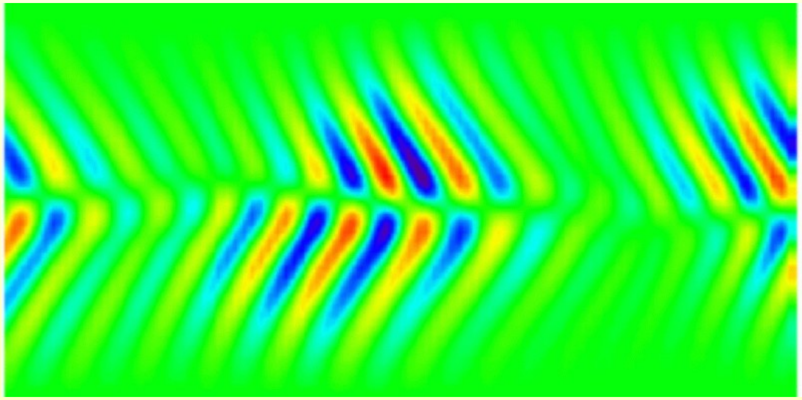
Mean field dynamo theory is very user friendly.

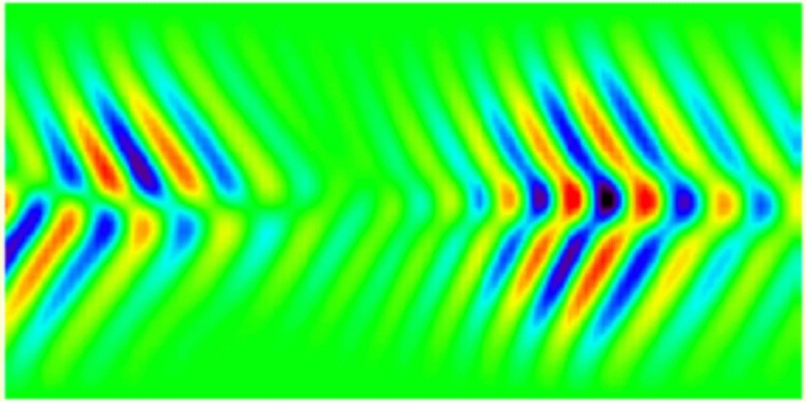
$$\frac{\partial B_0}{\partial t} = \nabla \times (U_0 \times B_0) + \nabla \times (\alpha B_0) + (\eta + \beta) \nabla^2 B_0.$$

For example, Cowling's theorem does not apply to the *mean* induction equation – allows axisymmetric solutions.

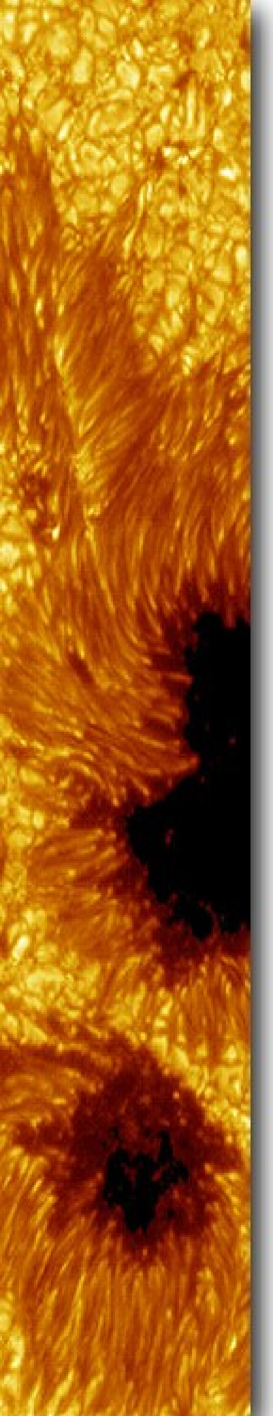
With a judicial choice of α and β (and differential rotation ω) it is possible to reproduce a whole range of observed astrophysical magnetic fields.

e.g. butterfly diagrams for dipolar and quadrupolar fields:



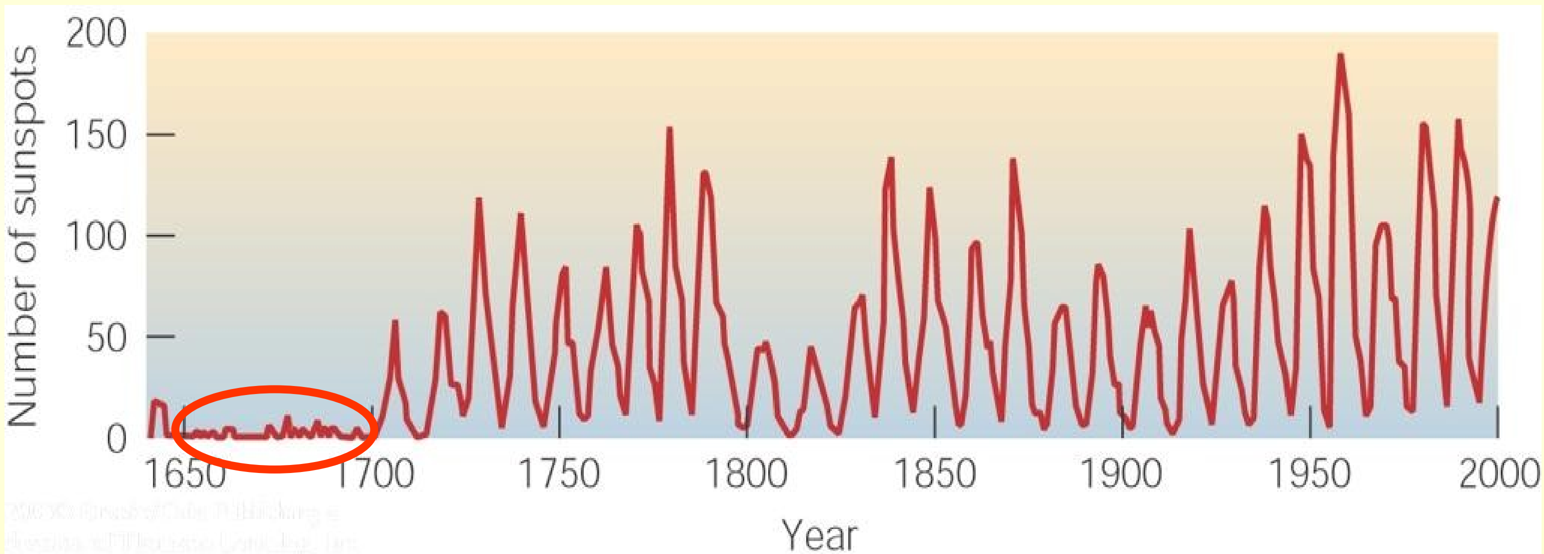


(Tobias 1996)



The Maunder Minimum

The sun spot number also fluctuates on much longer time scales:



Historical data indicate a very quiet phase of the sun, ~ 1650 – 1700: The **Maunder Minimum**

Little Ice Age (1650-1700)

- During a period that lasted approximately 50 years from the mid 1650s to the early 1700, the temperatures in northern Europe had their lowest values for the past millennium, with winter temperatures being on average 1 to 2 degrees colder than in later periods.
- This period has been called the ***Little Ice Age***. At this time, canals in Holland routinely froze solid, glaciers advanced in the Alps, and sea-ice increased so much that no open water was present in any direction around Iceland in 1695.

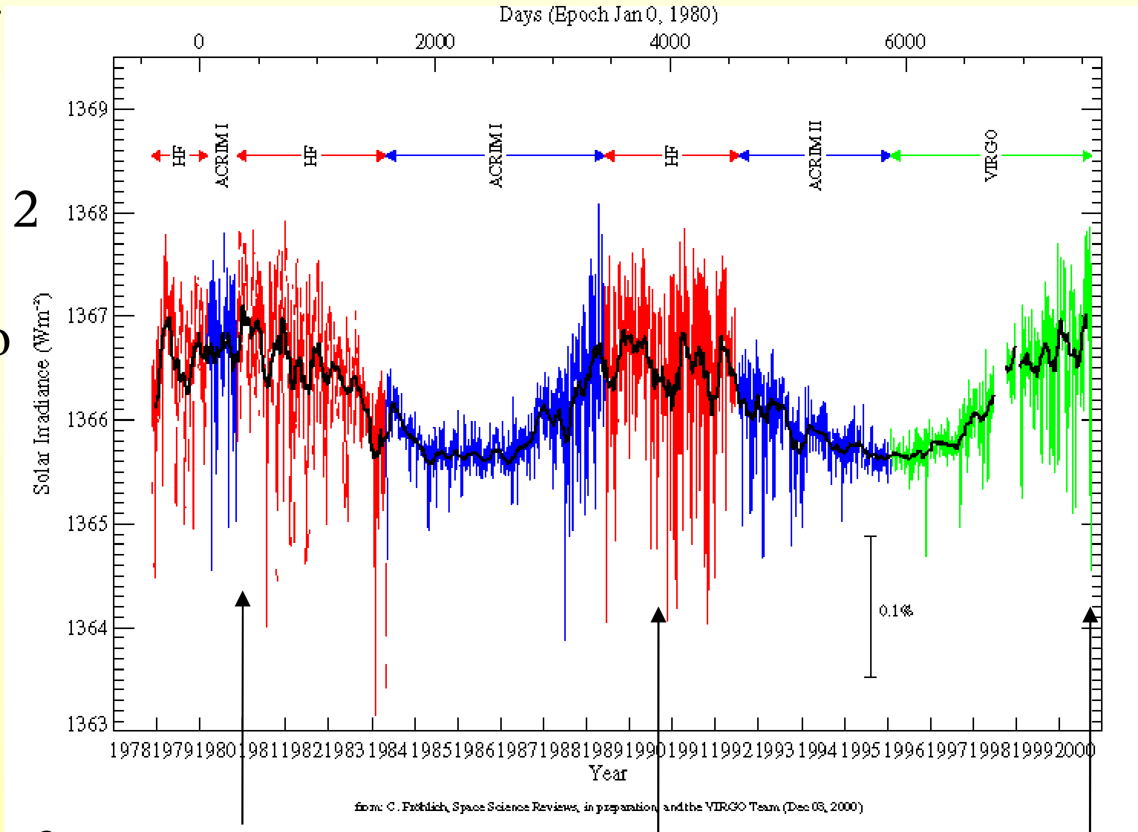
Aert van der Neer, Dutch, 1603-1688
Winter Scene with Frozen Canal

Solar Irradiance Variations

Modern measurements showed that the solar constant is really not a constant. The energy output of the Sun is modulated by the magnetic activity.

- Solar irradiance is **higher** when the surface **magnetic field is stronger** (when ther are more sunspots)…
- The amplitude of the solar irradiance variation is about 2 W/m^2, or about 0.1%.
- This variation is too weak to cause climate change.
- But, if solar magnetic activities was significantly reduced or enhanced for a long period of time, it can change the climate of the Earth…for example, did the Sun caused the ***Little Ice Age?***

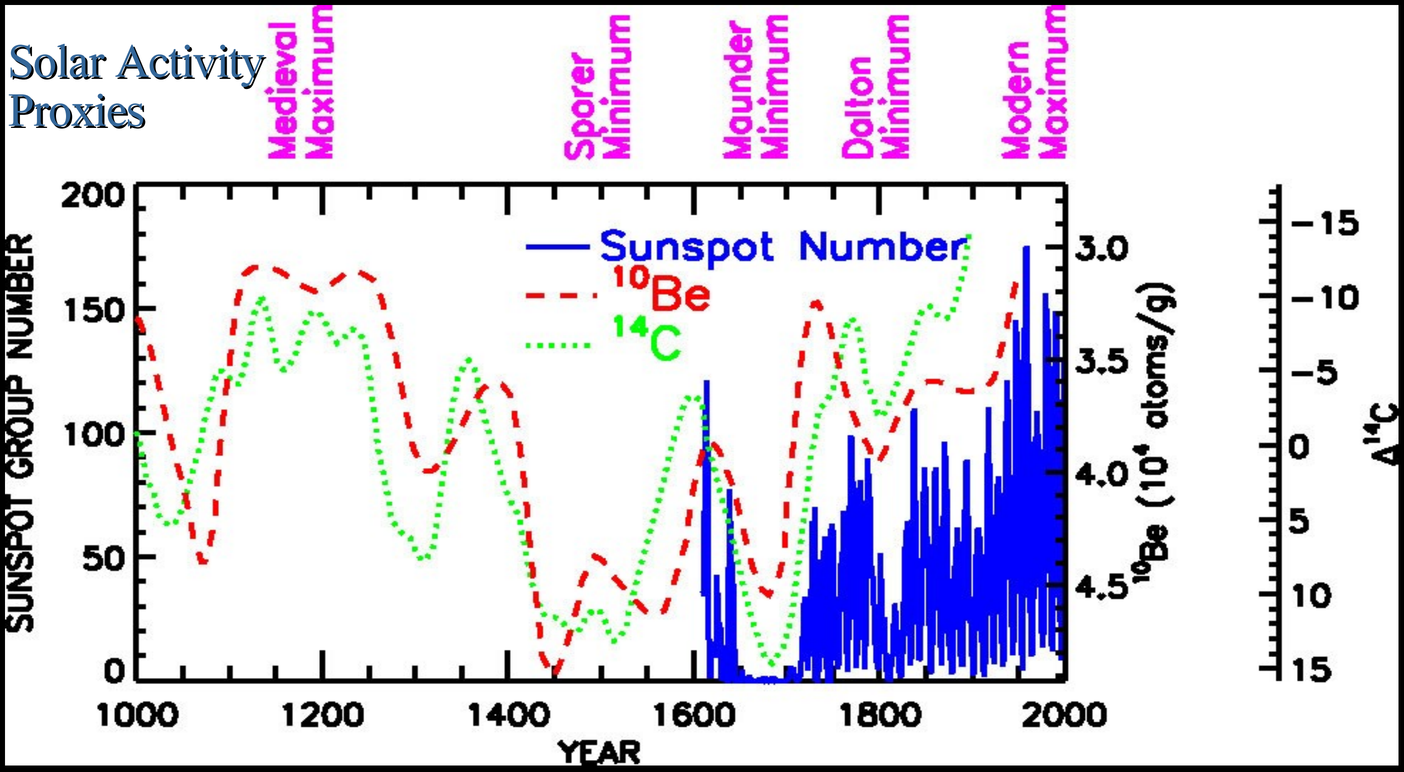
Solar constant measurements from several satellite experiments

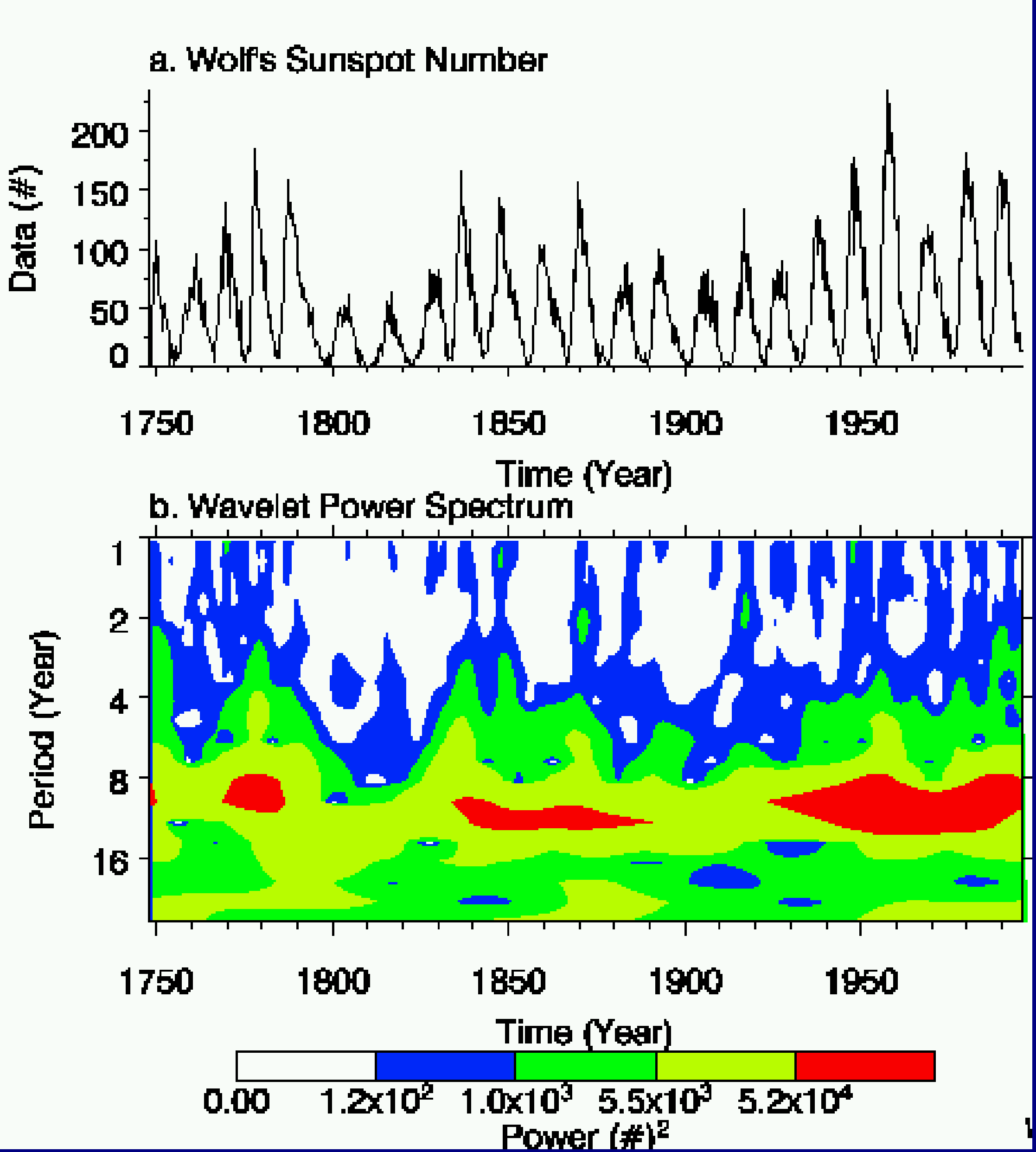


Sunspot Maximum

Long-Term Solar Variability

Solar activity proxies -- cosmogenic isotopes in tree rings and ice cores (below), geomagnetic activity, and the range of variability in Sun-like stars (right) -- suggest that long-term fluctuations in solar activity exceed the range of contemporary cycles.





The SUN

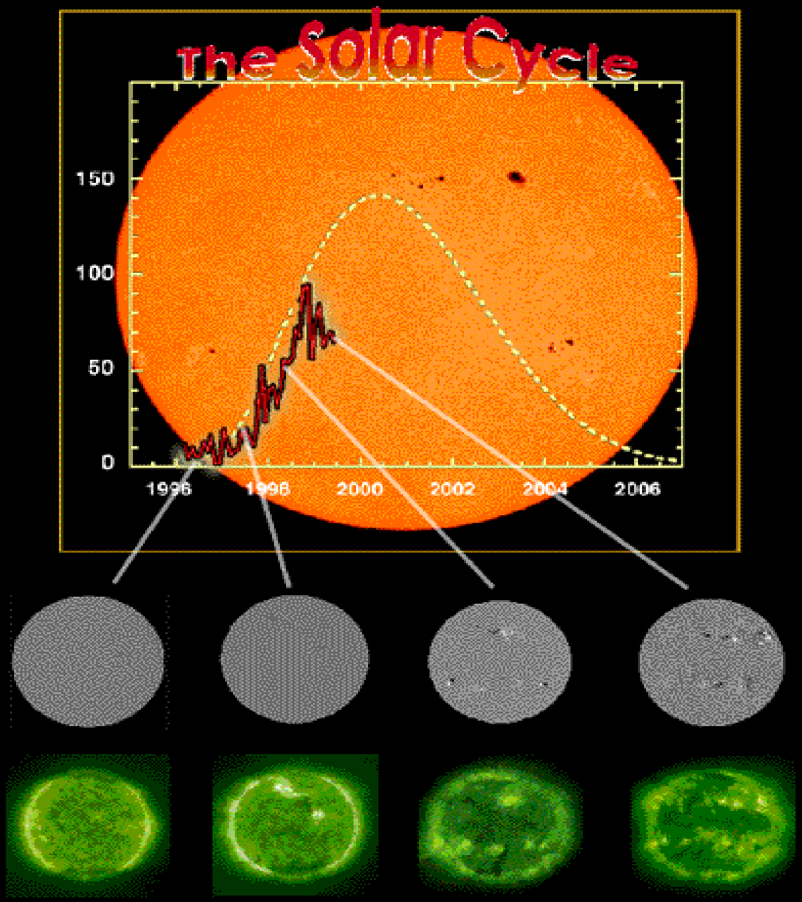
POWER SPECTRUM

Wavelet analysis

how the Fourier periods (y) vary in time (x).

http://ion.researchsystems.com/IONScript/wavelet/

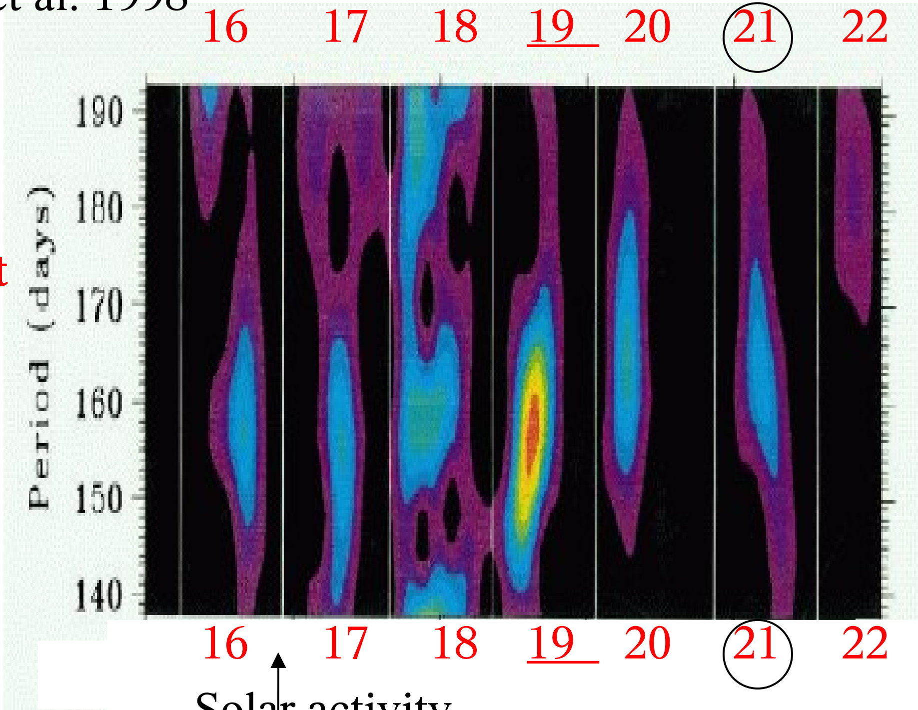
www.chabotspace.org/vsc/solar/thesun/images/solcycle-images-soho.gif



Research for short periodicities :
Timing analysis of **daily** sunspot areas/ group sunspot numbers

Oliver et al. 1998

Daily sunspot areas



Solar activity minimum

1874 - 1993

seven 11-yr cycles: from 16^{th} to 22th

Daily sunspot areas

Sunspot area:
The area of a sunspot is measured in a fraction (millionth) of the Sun's visible hemisphere.

Rieger Period : 154 days